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Approximation Theory:
A Volume Dedicated to Blagovest Sendov
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Blagovest Sendov and his Contribution to Approximation Theory

There is hardly a person who could give a complete and true picture of Sendov and his diverse numerous deeds, of his undertakings and achievements. Gifted with many of the attributes of a successful leader, a man of will-power, unusual intuition and ingenuity, he was naturally involved in activities far beyond the field of education and science. His impact on the political and intellectual life in Bulgaria in the last forty years is undoubtedly significant. When one looks over the record of his activities, even only in mathematics and mathematical education, it is hard to see how one person could manage his time in such a way to accomplish so much. Now, in his seventies, he continues astonishing even his closest collaborators with new ideas, plans and dreams, and the energy to realize them. Thus, any evaluation of his work at this moment would seem premature and fragmentary. I do not feel equal to this task. I shall just recall some of his profoundly original results in approximation theory - the subject he loves most.

Sendov was born on February 8, 1932 in Asenovgrad, a small town situated in the south part of Bulgaria. He got his Ph Degree from Sofia University in 1964. Coming from a well-being family, he had to face many restrictions imposed by the communist regime in Bulgaria after the World War II. He overcame all those difficulties due to his typical everlasting feature - never to give up. Being rejected a permit to study from the local authorities in his native town, he moved to Sofia and worked as a street cleaner 2 years in order to secure such a permit as a representative of the "labor class".

As a student at Sofia University, Sendov distinguished himself as a very talented mathematician, a problem solver, and an active participant in the student seminar run by Professor Tagamlitzki. His first paper ("On a class of regular-monotone functions", *Dokl. AN SSSR* **110**, 1 (1956), 27–30) resulted from work done as an undergraduate. It was presented to the Doklady of the Soviet Academy of Sciences by Professor Bernstein. As most of the students that time Sendov was fascinated by the talent and the charming personality of Professor Nikola Obrechkoff. Sendov graduated one year earlier and applied for a PhD study. Although he passed the examinations excellently, he was not admitted because of political reasons. He left the university and worked one year as a mathematics teacher in a small village. In fact, due to the intercession of Obrechkoff and other professors from Sofia University, he was finally offered a position of Assistant Professor at the Department of Mathematics.

Sendov had the good fortune to visit Moscow State University in the early sixties, the time when the approximation theory was opening new exciting fields as n-widths, ϵ-entropy, complexity, optimal quadratures, most of them arising from the work and ideas of Andrei Nikolaevich Kolmogorov. The seminars of Kolmogorov, Nikolski, Stechkin, Menshov had been gathering brilliant young mathematicians. Bernstein was still active in mathematical research. This period, brimming over with exciting mathematical events, the contacts with outstanding mathematicians, and the dominating influence of Kolmogorov had marked Sendov's research interests for years to come. Namely here, in Moscow, working on a problem posed by Kolmogorov about the entropy of the space of continuous functions, he realized the importance of the Hausdorff distance for approximation problems.

As is known, for any pair F, G of bounded point sets in the plane, the *Hausdorff distance* $r(F, G)$ between F and G is defined by

$$r(F,G) := \max \left\{ \max_{\mathbf{x}\in F} \min_{\mathbf{y}\in G} d(\mathbf{x},\mathbf{y}), \max_{\mathbf{x}\in G} \min_{\mathbf{y}\in F} d(\mathbf{x},\mathbf{y}) \right\},$$

where the distance $d(\mathbf{x}, \mathbf{y})$ between the points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is

$$d(\mathbf{x},\mathbf{y}) := \max \{|x_1 - y_1|, |x_2 - y_2|\}.$$

This metric is very natural and suitable for approximation of curves and point sets in the plane. To any bounded real-valued function $f(x)$ one can put in correspondence the point set $\bar{f}$ of its *completed graph*, that is, the intersection of all closed and convex, with respect to y axis, points sets in the plane which contain the graph of the function $f(x)$. Then the distance between any two bounded functions on a given interval Δ can be defined as a Hausdorff distance between their completed graphs, i.e., $r(f, g) := r(\bar{f}, \bar{g})$. This way, spaces of non-smooth, discontinuous or set-valued functions can be equipped with an appropriate metric of Hausdorff type. The first important contribution of Sendov to approximation theory was the development of the theory and adequate technique for Hausdorff approximation. In his PhD Thesis he showed that every bounded function on Δ can be approximated by polynomials of degree n with a rate $\frac{\log n}{n}$. The crucial point in the proof was the construction of a polynomial approximation of this rate on $[-1, 1]$ to the particular functions $s(x)$ (the *jump function*),

$$s(x) := \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

and the *delta function* $\delta(x)$ (which equals 1 at 0 and is zero at all other points). He showed also that these function can not be approximated better than *const.*$\frac{\log n}{n}$ with respect to the Hausdorff distance.

This new approach to approximation of functions reveals further properties even of classical and well-studied approximation schemes. For example, it is a basic fact in analysis that any continuous function can be approximated by the Bernstein polynomials

$$B_n(f;x) := \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$$

with respect to the uniform distance. But what is the behaviour of the sequence $\{B_n(f;x)\}$ for functions with discontinuities? In one of his earlier papers Sendov showed that for any bounded function f on $[0,1]$ the Bernstein polynomials $B_n(f;x)$ approach f with respect to the Hausdorff distance as n tends to infinity.

It was a nice idea to introduce a parameter α in the original definition of the Hausdorff distance between functions and get as a limit case (as $\alpha \to 0$) the uniform distance. Roughly speaking, the Hausdorff neighborhood with a parameter α of a given function is the area swept by a rectangle with a center (i.e., the intersection of its diagonals) traversing the graph of the function, and having sides in a ratio α. For $\alpha = 0$ the rectangle degenerates into a segment parallel to y-axis and one arrives at the definition of a neighborhood with respect to the uniform distance. In this way many results in the theory of uniform approximations can be obtained as particular cases of their Hausdorff analogs.

The next generalization of the classical Jackson theorem was established in a paper Sendov wrote jointly with Vasil Popov for the *Journal of Approximation Theory* **9** (1973), 102–111:

There exists an absolute constant C such that for every bounded function f on Δ and any $\alpha > 0$, the best approximation $E(H_n, \Delta, \alpha; f)$ of f on Δ by polynomials of degree n with respect to the Hausdorff distance with a parameter α satisfies

$$E(H_n, \Delta, \alpha; f) \leq C\,\omega(f; n^{-1})\,\frac{\ln(e + \alpha n\omega(f; n^{-1}))}{1 + \alpha n\omega(f; n^{-1})}.$$

In case f is continuous, letting α tend to zero we obtain the estimate $E_n(f) \leq C\omega_n(f; n^{-1})$ for the uniform approximation of f by polynomials from H_n (i.e., the space of polynomials of degree $\leq n$), which is the Jackson result.

The interest to Sendov's work on Hausdorff approximation was quite strong, in particular in the former Soviet Union, and shortly after the publication of his first papers he received an invitation to write a survey on this topic for the celebrated Russian periodical *Uspehi Matematicheskih Nauk*. Sendov's paper appeared in 1969. A detailed account on the results in this field have been presented in Sendov's first monograph "Hausdorff Approximations", BAN, Sofia, 1979 [in Russian], subsequently translated in English.

Sendov devoted considerable effort and time during his most productive years to the Hausdorff approximation. He wrote (alone or in collaboration with others) more than 70 papers on this topic. After his return from Moscow in 1961 he organized a seminar on approximation theory at Sofia University and introduced to this new way of approximation his first students, among them, the most distinguished one - Vasil Popov. Soon the seminar turned into a center of excellence in approximation theory on international scale. Visitors from many countries had delivered talks there. In fact, this seminar and the inspiring work of Sendov led to the formation of the Bulgarian school in approximation theory. Over the years, some of the best students in mathematics at Sofia University, as well as young researchers from other fields, used to attend the seminar attracted by the creative, stimulating atmosphere there, by the numerous challenging problems Sendov was stating regularly. The ability of Sendov to suggest the right problems, to explain them clearly, to outline ways in which they might be attacked, and, which is most important, to fire the students with the excitement of discovery, is really amazing.

One of the problems posed by Sendov that time and staying open so long is:

Prove that every polynomial of degree n can be approximated with respect to the Hausdorff distance by a polynomial of degree $n-1$ with a rate $O(n^{-1})$

As a result of the work on Hausdorff distance Sendov introduced a new characteristic of the functions called *τ-modulus*. For each bounded function f on $[a,b]$, the averaged modulus of smoothness of order k (i.e., the τ-modulus) is defined by

$$\tau_k(f;\delta) := \left\{ \frac{1}{b-a} \int_a^b \Big(\omega_k(f,x;\delta)\Big)^p \, dx \right\}^{1/p},$$

where

$$\omega_k(f,x;\delta) := \sup \left\{ |\Delta_h^k f(t)| \;:\; t, t+kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}] \cap [a,b] \right\}.$$

This type of moduli is suitable for estimating the error of approximation processes under weaker restrictions on the smoothness of the functions considered. Sendov and his collaborators generalized many of the basic results in numerical analysis obtaining estimates in terms of the τ-modulus. This aspect of his scientific interest was covered by the monograph "The Averaged Moduli of Smoothness", BAN, 1983, written jointly with Vasil Popov. Five years later the book was translated and published in Russian (by Mir, Moskow, 1988) and in English (by John Wiley & Sons, New York 1988).

One of the most exciting enterprise for Sendov, as well as for his students and colleagues, was his epic struggle with the Whitney constant. In 1957 Whitney showed that for each natural n, there is a constant W_n such that

$$E_{n-1}(f) \leq W_n \, \omega_n(f, \frac{1}{n}).$$

The initial bounds for W_n have been of order n^n. There was a certain belief among the approximation specialists that the exact order is likely c^n. Sendov jumped to this nice classical problem with his typical passion, devotion and persistence. He delights in discussing mathematics and is always ready to share his ideas with anyone who cares to listen. At this period, any discussion with him would finally lead to Whitney constant. He was ghosted by the idea to find the exact order. His anxiety and interest to this problem was transferred to his pupils and collaborators. Some of them were helping him to test various hypothesis by computer. Then, one day, he announced his striking conjecture that the Whitney constant does not depend on n, moreover, the constant is 1. The first important breakthrough to the proof of this unbelievable conjecture was done by Milko Takev, a PhD student of Sendov. He wrote a manuscript in which he described an ingenious approach to the estimation of the error in Lagrange interpolation with equidistant nodes which leads to estimation of the Whitney constant by c^n.

This bound was subsequently diminished repeatedly in a short period by Sendov and his collaborators, and was reduced to $C \ln n$. Finally, Sendov derived the following fine representation of the remainder in the interpolation of any integrable function f at the equidistant nodes $h, 2h, \ldots, nh$, $h = 1/(n+1)$, namely

$$\begin{aligned} f(x) - P_{n-1}(f;x) &= \Delta_h^n f(0) \ell_{n0}\left(\frac{x}{h}\right) + \varphi_n(f;x) - \sum_{j=0}^{n} \varphi_n(f;jh)\ell_{nj}\left(\frac{x}{h}\right) \\ &+ \sum_{j=0}^{n} \frac{1}{h} \int_0^{\tau} \varphi_n(f;jh+\tau)\ell'_{nj}\left(\frac{x-\tau}{h}\right) d\tau, \end{aligned}$$

where

$$\varphi_n(f;x) := \frac{(-1)^{n-\nu}}{h\binom{n}{\nu}} \int_0^h \Delta_\tau^n f(x-\nu\tau)\, d\tau, \quad x = \nu h + \tau,$$

and ℓ_{nj} are the basic Lagrange polynomials associated with the aforementioned nodes. Then, using a clever estimation of the Lagrange basic polynomials, he obtained

$$E_{n-1}(f) \le 6\,\omega_n(f, \frac{1}{n+1}).$$

The conjecture that $W(n) = 1$ is still open.

Gifted with a strong intuition, amazing insight and ability to catch the essence of a mathematical result, Sendov often surprises his colleagues with a certain non-standard point of view or an original treatment of questions in a field that is new for him. Sometime he could come up with an interesting problem just glancing over the first few pages of a book, or article. More than 40 years ago he started his university career as an assistant of Professor Obrechkoff, whose most favorite subject was the geometry of polynomials. In his initial attempt to learn more about the work there, Sendov arrived at a remarkable problem, trying to find a counterpart of Rolle's theorem in the complex case. He asked the following:

Assume that $P(z)$ is an algebraic polynomial whose all zeros lie in the unit disk. Is it true that every disk with a radius 1 *and centered at a zero of P contains a zero of the derivative P'?*

Now more than 80 papers are published on this question and it is still unresolved. Recently Sendov wrote a survey on this problem in *East Journal on Approximations* **7**, 2 (2001), 123–178, where one can see the recent progress. The interest to this problem is growing and it has all chances to become one of the famous problems in the geometry of polynomials, because: the formulation is extremely simple and appealing; the first impression is that this is just a problem from a student exam in complex analysis; the delicate and complicated relationship between the zeros of a polynomial and the zeros of its derivative resist so far any treatment with the known tools of analysis.

Another question raised by Sendov provoked a number of publications and opened a new topic, which is referred to as *parametric approximation.* The standard approximation problem is to find a polynomial P of a given degree n that approximate best a given function $f(x)$. Sendov proposed to look for a pair of polynomials (P, Q) that minimize the error $\|f(Q(\cdot)) - P(\cdot)\|$ on $[-1, 1]$. The polynomial Q can be restricted to the set of monotone polynomials such that $Q(-1) = -1, Q(1) = 1$. He showed in a very elegant way that the rate of uniform parametric approximation of the function $|x|$ is $(3+2\sqrt{2})^{-n}$ [*Annuaire Univ. Sofia, Faculte Math.* **64** (1971), 237–247. Note that according to the famous result due to Donald Newman, the exact order of the rational approximation to $|x|$ is of order $e^{-c\sqrt{n}}$, i.e., much weaker than the parametric one. Moreover, Sendov (in the case of even n) and others (in the case of odd n) found closed form expressions, in terms of the Tchebycheff polynomials, for the pair of best uniform parametric approximation to $|x|$, while the polynomial and the rational function of best approximation to $|x|$ are not known explicitly.

Apart from his constant interest to Hausdorff approximation Sendov always keeps up with the new topics in approximation theory. His recent interest is in the multiresolution analysis, compression and image processing, where he demonstrates his own original way to these new developments.

It is amazing how Sendov combines such an intensive research with an active educational, political and administrative position. Over the years he has served as a Rector of the Sofia University, President of the Bulgarian Academy of Sciences, Chairman of the Bulgarian Parliament, Member of the Parliament in the last 20 years. He acted as a President of the International Association of the Universities, President of the International Federation for Information Processing (IFIP), Member of the Executive Committee and Board of Directors of the International Foundation for the Survival and Development of Humanity.

Sendov is a tireless popularizer of the mathematics and computers. He was a member of the team that built the first Bulgarian computer. He wrote dozens of articles and gave numerous lectures in Bulgarian schools, meetings and conferences, participated in radio and TV discussions about mathematics and the importance of computer education.

Taking high administrative positions Sendov had never hesitated to take responsibility for difficult decisions in favor of the science and the scientists, often in conflict with the formal understanding and requirements of the ruling parties. There are dozens of instances when he had helped young talented mathematicians in their professional career.

I join his many friends in wishing him a good health and continued joy of mathematical discovery.

Borislav Bojanov