On V. K. Dzjadyk's Investigations in Approximation Theory

I. A. Shevchuk
National Taras Shevchenko
University of Kyiv

This lecture was delivered at the conference *Extremal Problems and Approximation* in honor of the 70th birthday of V. M. Tikhomirov, and was held in Moscow on December 16–18, 2004. The lecture was presented by Prof. V. N. Konovalov and the author. Prof. V. N. Konovalov first gave a survey of the papers [1-4] which are all devoted to Prof. V. K. Dzjadyk, following which the author spoke about the following.

I. Constructive Theory on $[-1,1]$**.** Let

$$\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}. \tag{1}$$

In 1956 V. K. Dzjadyk proved that for each algebraic polynomial P_n of degree n and each $s \in \mathbb{R}$:

$$\left\| P_n' \rho_n^{s+1} \right\|_{C[-1,1]} \le c(s) \left\| P_n \rho_n^s \right\|_{C[-1,1]}. \tag{2}$$

If $s=0$, then (2) is the Markov-Bernstein inequality. The inequality (2) made possible a solution to the following problem of constructive characterization.

Let H^α, $\alpha > 0$, be the Nikolskii subspace of functions, and $f \in C[-1,1]$. Then,

$$f \in H^\alpha \quad \Longleftrightarrow \quad \inf_{P_n} \left\| \frac{f - P_n}{\rho_n^\alpha} \right\|_{C[-1,1]} = O(1), \quad n \to \infty.$$

The implication "$\Longleftarrow$" follows from Dzjadyk's inverse theorem, and "$\Longrightarrow$" follows from the classical Nikolskii type pointwise estimates proved by A. F. Timan (k=1, 1951), V. K. Dzjadyk (k=2, 1958), G. Freud ($k=2$, 1959) and Yu. A. Brudnyi ($k>2$, 1968):
If $f \in C^{(r)}[-1,1]$, then, for all $n \ge k+r-1$, there exists an algebraic polynomial P_n of degree n such that

$$|f(x) - P_n(x)| \le c(k,r)\rho_n^r(x)\omega_k(\rho_n(x), f^{(r)}), \qquad x \in [-1,1], \tag{3}$$

where ω_k is the modulus of continuity of order k.

With $E_n(f)_{C[-1,1]}$ the error in the best uniform approximation to f by polynomials of degree n, this estimate readily implies a Jackson-Stechkin type estimate: *if $f \in C^{(r)}[-1,1]$, then, for each $n \ge k+r-1$,*

$$E_n(f)_{C[-1,1]} \le \frac{c(k,r)}{n^r}\omega_k\left(\frac{1}{n}, f^{(r)}\right).$$

II. Constructive Theory on the Sets in $\mathbb{C}$. What is ρ_n defined by (1)? For $x \in [-1,1]$, let $d_n(x)$ be the distance from $x \in [-1,1]$ to the n-th level line of the set $[-1,1] \subset \mathbb{C}$. Then

$$\frac{1}{6}\rho_n(x) < d_n(x) < \rho_n(x).$$

Recall that the n-th level line of a continuum $G \subset \mathbb{C}$ is a curve defined by $\Gamma_n := \{z \in \mathbb{C} : |\Phi(z)| = 1 + \frac{1}{n}\}$, where Φ is the Riemann function, mapping the exterior of G onto the exterior of the unit disk, and normed by $\Phi'(\infty) > 0$.

In 1958 S. M. Nikolskii conjectured that a constructive characterization on closed sets $G \subset \mathbb{C}$ may be obtained using the distance $\rho_n(z)$ from points $z \in \partial G$ to the n-th level line of the set G.

V. K. Dzjadyk was the first to establish such a constructive theory on sets in $\mathbb{C}$. In 1959 he proved the inequality

$$\left\|P'_n \rho_n^{s+1}\right\|_{C(\partial G)} \leq c(s,G) \left\|P_n \rho_n^{s}\right\|_{C(\partial G)} \tag{2'}$$

for any set G with piecewise smooth boundary and some other restrictions, and proved the inverse theorem.

In 1970 N. A. Lebedev and P. M. Tamrazov generalized the inequality (2′) and the inverse theorem to sets of a very general structure. These ideas permitted P. M. Tamrazov to obtain so-called contour-solid theorems. This latter direction was developed by P. M. Tamrazov and his pupils as well as by F. W. Gehring, W. K. Heyman, A. Hinkkanen L. A. Rubel, A. L. Shields, B. A. Taylor, and others.

In the beginning of the 1960's, V. K. Dzjadyk proved a direct theorem. The theory of direct theorems for the sets $G \subset \mathbb{C}$ was developed by V. K. Dzjadyk and his pupils and V. A. Andrievskii, V. I. Belyi, N. A. Lebedev, E. Saff, N. A. Shirokov and many others.

The main tool used in the proofs of the direct theorems is the so-called Dzjadyk polynomial kernel $D_n(\zeta, z, R)$, $\zeta \in \partial G, z \in \mathbb{C}, R > 1$. For a set G with piecewise smooth boundary and some minor restrictions, Dzjadyk's polynomial kernel $D_n(\zeta, z) := D_n(\zeta, z, 1 + 1/n)$ has the following properties.

Theorem D. *For each fixed $m \in \mathbb{N}$ and any $n \in \mathbb{N}$, there exists a Dzjadyk polynomial kernel $D_n(\zeta, z)$, having the properties*
a)

$$D_n(\zeta, z) = \sum_{k=0}^{n-1} a_k(\zeta) z^k,$$

where a_k,, $k = 0, ..., n-1$, are functions continuous on ∂G;
b) for each $p = 0, 1, \ldots, m$, $\zeta \in \partial G$ and $z \in \overline{G}$, we have

$$\left|\frac{\partial^p}{\partial z^p}\left(\frac{1}{\zeta - z} - D_n(\zeta, z)\right)\right| \leq \frac{C(G,m)}{|\zeta - z|^{p+1}} \left(\frac{\rho_n(z)}{|\zeta - z| + \rho_n(z)}\right)^m,$$

where $\zeta \neq z$*, and*

$$\left|\frac{\partial^p}{\partial z^p} D_n(\zeta, z)\right| \leq \frac{C(G,m)}{(|\zeta - z| + \rho_n(z))^{p+1}};$$

c)

$$\frac{1}{2\pi i} \int_{\partial G} D_n(\zeta, z) d\zeta = 1,$$

d) if P_m *is an algebraic polynomial of degree* $\leq m$*, then, for all* $z \in \overline{G}$ *and* $p = 0, 1, 2, \ldots,$

$$\left|P_m^{(p)}(z) - \frac{1}{2\pi i} \int_{\partial G} P_m(\zeta) \frac{\partial^p}{\partial z^p} D_n(\zeta, z) d\zeta\right| \leq \frac{C}{n^m} \max_{j \geq p} |P_m^{(j)}(z)|,$$

where $C = C(G, m)$*.*

III. Ditzian-Totik Moduli of Smoothness. In 1968, V. K. Dzjadyk and G. A. Alibekov obtained a *uniform* constructive characterization of functions on the sets $G \subset \mathbb{C}$. To this end, they characterized the smoothness of a function at a point $z \in \partial G$. Further research in these directions were undertaken by Yu. I. Volkov, E. M. Dynkin, V. A. Andrievskii and others. For the interval $x \in [-1, 1]$, corresponding definitions of smoothness (depending on a point) were given by M. K. Potapov, K. I. Babenko, Bl. Sendov, K. Ivanov (in the 80's). Now, for $[-1, 1]$, the Ditzian-Totik moduli of smoothness are used in such investigations.

IV. Interpolation at the End-points. S. A. Telyakovskii, A. I. Gopengauz, R. A. DeVore, H. H. Gonska, E. Hinneman, X. M. Yu, D. Leviatan, H-J. Wenz and others proved that, sometimes,

$$\rho_n(x) = \frac{1}{n^2} + \frac{\sqrt{1 - x^2}}{n}$$

in (3) may be replaced by

$$\varphi_n(x) := \frac{\sqrt{1 - x^2}}{n},$$

and sometimes it may not.

Proposition. *If* $f \in C^{(r)}[-1, 1]$*, then for all* $n > k + r$ *there exists an algebraic polynomial* P_n *of degree at most* n *for which*

$$|f(x) - P_n(x)| \leq c(k, r) \varphi_n^r(x) \omega_k(\varphi_n(x), f^{(r)}), \qquad x \in [-1, 1],$$

if and only if the pair (k, r) *is marked with a '+' in the following display.*

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
3	+	+	+	+	+	+	−	$\cdots$
2	+	+	+	+	+	−	−	$\cdots$
1	+	+	+	+	−	−	−	$\cdots$
0		+	+	−	−	−	−	$\cdots$
	0	1	2	3	4	5	6	k

V. Shape-preserving Approximation. In 1985 R. A. DeVore and X. M. Yu proved: *if $f \in C[-1,1]$ is a monotone function on $[-1,1]$, then for each n there is a monotone algebraic polynomial P_n of degree at most n for which*

$$|f(x) - P_n(x)| \le c\omega_2(\varphi_n(x), f), \quad x \in [-1,1].$$

To prove this result, they used the kernel

$$K_n(y,x) := \int_{\beta-\alpha}^{\beta+\alpha} J_n(t)\,dt,$$

where $\beta := \arccos x$, $\alpha := \arccos y$, and J_n is a Jackson-type kernel. It turns out that

$$\frac{\partial}{\partial x} K_n(y,x) = \frac{1}{\pi} \lim_{R\to 1} \operatorname{Im} D_n(y,x,R),$$

where D_n is Dzjadyk's polynomial kernel for the set $G = [-1,1]$!

References

[1] Kornejchuk N. P., Nikol'skij S. M., Shevchuk I. A., Vladislav Kirillovich Dzyadyk, Russ. Math. Surv. 34, No. 4, 213–221 (1979).

[2] Belyj V. I., Golub A. P., Shevchuk I. A., Dzyadyk's research on the theory of approximation of functions of a complex variable, Ukr. Math. J. 41, No. 4, 384–395 (1989); translation from Ukr. Mat. Zh. 41, No. 4, 441–454 (1989).

[3] Nikol'skij S. M., Korneichuk N. P., Shevchuk I. A., Vladislav Kirillovich Dzyadyk (on the occasion of his seventieth birthday), Uspekhi Mat. Nauk 44, No. 4(268), 243–244 (1989); translation in Russian Math. Surveys 44, No. 4, 213–215 (1989).

[4] Stepanets A. I., V. K. Dzyadyk's research on the theory of approximating periodic functions. (Russian) Ukrain. Mat. Zh. 41, No. 4, 436–441, (1989); translation in Ukrainian Math. J. 41, No. 4, 379–383 (1989).

shevchuk@univ.kiev.ua