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ARCHIVES:

ON THE LOWER BOUND OF THE MAXIMUM OF CERTAIN POLYNOMIALS*

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Introduction

The next theorem of A. Markoff[1] is well-known:
Let $f(z)$ be an arbitrary polynomial of degree n for which

$$|f(z)| \leqq 1 \tag{1}$$

in the interval $-1 \leqq z \leqq 1$; then also the inequality

$$|f'(z)| \leqq n^2 \tag{2}$$

holds in this interval, and equality can be attained only at $z = \pm 1$. The polynomial yielding the extreme value is $\pm\cos(n \arccos z)$.

The above theorem can also be formulated as follows: If $f(z)$ is a polynomial of degree n, then

$$\frac{\max\limits_{-1 \leqq z \leqq 1} |f'(z)|}{\max\limits_{-1 \leqq z \leqq 1} |f(z)|} \leqq n^2. \tag{3}$$

The value of the derivative in the inner points of the interval was estimated by S. Bernstein[2]: *If* (1) *holds in the interval $-1 \leqq z \leqq 1$, then in the same interval we have*

$$|f'(z)| \leqq \frac{n}{\sqrt{1-z^2}}. \tag{4}$$

*This paper is a complete translation in English from Hungarian of the paper "Bizonyos polinomok maximumának alsó korlátjáról" by J. Erőd. The original paper has appeared in: *Mat. Fiz. Lapok* **46** (1939), pp. 58–82.

[1] *Über ein Problem von D. J. Mendeleeff.* (Russian, with German abstract.) St. Petersburg Academy Publ. **62** (1889), 1–24.

[2] *Sur l'ordre de le meilleure approximation des functions continues par des polynômes de degré donné.* Mém. publ. par la Cl. des Sc. de l'Acad. de Belgique **4** (1912).

Similarly to the above, a bound of the derivative of polynomials can be looked for, when inequality (1) is valid for a certain set of points in the complex plane. The first result in this field is due to to M. Riesz[3]:

If (1) *is valid on the circumference of the circle* $|z| \leqq 1$, *then in this domain*

$$|f'(z)| \leqq n \tag{5}$$

is valid, and equality holds only for polynomials of the form $\alpha \cdot z^n$, *where* $|\alpha| = 1$. That is, for every polynomial of degree n,

$$\frac{\max\limits_{|z|\leq 1} |f'(z)|}{\max\limits_{|z|\leq 1} |f(z)|} \leqq n. \tag{6}$$

W. E. Sewell[4] generalized Riesz' theorem for elliptic domains: *If* (1) *is valid on the boundary of an ellipse having axes* $(-1, 1)$ *and* $(-ai, ai)$, $0 \leqq a \leqq 1$, *then*

$$|f'(z)| \leqq \frac{n}{\sqrt{1 + a^2 - |z|^2}}. \tag{7}$$

G. Szegő[5] extended Markoff's theorem on closed domains M bounded by Jordan arcs. According to him

$$\frac{|f'(z_0)|}{\max\limits_{z \in M} |f(z)|} \leqq c_1(M, z_0) \cdot n^\alpha \tag{8}$$

holds *for polynomials of degree* n *if* z_0 *is a boundary point of* M, *and the points of* M *fall between two halflines drawn from* z_0, *and enclosing an angle* $\alpha \cdot \pi$ $(0 < \alpha \leqq 2)$. *Furthermore,*

$$\frac{|f'(z_0)|}{\max\limits_{z \in M} |f(z)|} \leqq c_2(M, z_0, \delta) \cdot (1 + \delta)^n \tag{9}$$

holds, where z_0 *is an arbitrary point of the plane.* Here c_1 and c_2 are constants independent of n, and $\delta > 0$ is arbitrary. Szegő's theorem is sharp with respect to the order of magnitude.

M. Fekete[6] showed that the roots of the polynomials, because of which (9) cannot be improved, lie on the set M. Pál Turán raised the question: What

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[3] *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome.* Jahresber d. deutschen Math. Vereinigung **23** (1914), 354–368.

[4] *On the polynomial derivate constant for an ellipse.* Amer. Math. Monthly **44** (1937), 577–578.

[5] *Über einen Satz von A. Markoff.* Math. Zeitschrift **23** (1925), 45–61.

[6] *Über den absoluten Betrag von Polynomen, welche auf einer Punktmenge gleichmässig beschränkt sind.* Math. Zeitschrift **26** (1927), 324–344.

can be stated conversely on the derivative of polynomials $f(z)$ of degree n whose roots lie on the set M, and which take on the value 1 in a point of the set M? The class of polynomials satisfying this condition will be denoted by $E(M)$. The question can also be posed in the following way: *Does there exist a lower bound for*

$$\frac{\max_{z\in M}|f'(z)|}{\max_{z\in M}|f(z)|}$$

over the set of polynomials $E(M)$*?*

Turán proved that if M_1 is the interval $(-1,+1)$, then for each polynomial $f(z)$ of degree n from $E(M_1)$ there exists a point ζ in M_1 at which

$$|f'(\zeta)| \geqq \frac{1}{6}\sqrt{n}, \tag{10}$$

and if the set M_2 is the unit circle, then for each polynomial $f(z)$ of $E(M_2)$ there exists a point $\zeta \in M_2$ for which

$$|f'(\zeta)| \geqq \frac{n}{2}; \tag{11}$$

moreover, the latter cannot be improved further. In other words, we have

$$\frac{\max\limits_{-1\leqq z\leqq 1}|f'(z)|}{\max\limits_{-1\leqq z\leqq 1}|f(z)|} \geqq \frac{1}{6}\sqrt{n}, \tag{12}$$

if $f(z) \in E(M_1)$, and

$$\frac{\max\limits_{|z|\leqq 1}|f'(z)|}{\max\limits_{|z|\leqq 1}|f(z)|} \geqq \frac{n}{2}, \tag{13}$$

if $f(z) \in E(M_2)$.

In § 1 Turán's result (see (10)) and its sharpening is discussed. We prove that *there exists a sequence tending to zero*

$$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$$

such that in case $f(z) \in E(M_1)$, *at a certain point* $-1 \leqq \zeta \leqq 1$,

$$|f'(\zeta)| \geqq \sqrt{\frac{n}{e+\varepsilon_n}} \tag{14}$$

holds, and (14) cannot be improved.

§ 2 contains a generalization of (10) and (11). *Let the domain M_3 be the ellipse as in Sewell's theorem. If $f(z) \in E(M_3)$, then there exists a point $\zeta \in M_3$ for which*

$$|f'(\zeta)| \geqq \max\left(\frac{n \cdot a}{2}, \frac{1}{7}\sqrt{n}\right). \tag{15}$$

In § 3 we study the question for what type of domains can solutions for Turán's problem be found with the methods in § 2.

Finally, in § 4 we give an application of the theorem in § 1 concerning a problem of P. Erdős. Let $f(z) \in E(M_1)$ and let $f(z)$ be convex (or concave) between its roots α and β. Then

$$|\alpha - \beta| \leqq \frac{c_n}{\sqrt{n}}, \tag{16}$$

where, by the proof by Erdős and Turán,

$$c_n \leqq 16, \tag{17}$$

while, according to the exact value found here,

$$\lim_{n=\infty} c_n = \sqrt{2}. \tag{18}$$

§ 1

a) Pál Turán proved the next theorem: *If all the roots of the polynomial $f(z)$ of degree n lie in the interval $(-1, 1)$ and in a certain point α*

$$|f(\alpha)| = 1 \tag{19}$$

holds, then there exists a point $-1 \leq \zeta \leq 1$ where

$$|f'(\zeta)| \geqq \frac{1}{6}\sqrt{n}. \tag{20}$$

Let namely $z_1, z_2, \dots, z_n$ be the roots of the polynomial $f(z)$, and let

$$-1 \leqq z_1 \leqq z_2 \leqq \cdots \leqq z_n \leqq 1. \tag{21}$$

We may assume that $f(z)$ and its derivatives are real for real values of z.

Let firstly $\alpha < z_1$, or $\alpha > z_n$, i.e. let the sign of $\alpha - z_i$ be the same for every i. Then

$$|f'(\alpha) = |f(\alpha)| \cdot \left|\sum_{i=1}^{n} \frac{1}{\alpha - z_i}\right| = \sum_{i=1}^{n} \frac{1}{|\alpha - z_i|} \geqq \sum_{i=1}^{n} \frac{1}{2} = \frac{n}{2} > \frac{\sqrt{n}}{6}; \tag{22}$$

Let secondly

$$z_i < \alpha < z_{i+1}. \tag{23}$$

Let a be a point of maximum of $|f(z)|$ in the interval (z_i, z_{i+1}). Then $|f(a)| \geqq 1$, so dividing by it the absolute value of the derivative becomes smaller everywhere. So we may assume that in the interval (z_i, z_{i+1})

$$|f(z)| \leqq 1. \tag{24}$$

We may also suppose that in the interval $\left(a - \frac{2}{\sqrt{n}}, a\right)$ the inequality

$$|f(z)| \geqq \frac{2}{3} \tag{25}$$

is valid, otherwise in a certain point β of this interval the estimate

$$|f'(\beta)| = \frac{|f(a) - f(z)|}{|a - z|} \geqq \frac{1 - \frac{2}{3}}{\frac{2}{\sqrt{n}}} = \frac{1}{6}\sqrt{n} \tag{26}$$

would hold. On the basis of (23) and (25) we see that $z_i < a - \frac{2}{\sqrt{n}} < a < z_{i+1}$.

We may assume further that in a point ζ of the interval $\left(a - \frac{2}{\sqrt{n}}, a\right)$

$$|f''(\zeta)| \leqq \frac{1}{12} n \tag{27}$$

holds, otherwise $f"(z)$ has a constant sign in this interval, a is the point of maximum, so

$$f'(a) = 0 \tag{28}$$

and we obtain

$$\begin{aligned} \left| f'\left(a - \frac{2}{\sqrt{n}}\right) \right| &= \left| \int\limits_{a-\frac{2}{\sqrt{n}}}^{a} f''(z)dz \right| \\ &= \int\limits_{a-\frac{2}{\sqrt{n}}}^{a} |f''(z)|dz \geqq \frac{1}{12} \cdot n \cdot \frac{2}{\sqrt{n}} = \frac{1}{6}\sqrt{n}. \end{aligned} \tag{29}$$

It is well-known that

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i}.$$

We differentiate this with respect to z at $z = \zeta$, where ζ is defined by (27):

$$\frac{|f(\zeta)f''(\zeta) - f'(\zeta)^2|}{f(\zeta)^2} = \sum_{i=1}^{n} \frac{1}{(\zeta - z_i)^2} \geqq \sum_{i=1}^{n} \frac{1}{2^2} = \frac{n}{4}. \tag{30}$$

On the basis of (25), (24) and (27) we obtain from (30) that

$$\begin{aligned} f'(\zeta)^2 &\geqq \left(\frac{2}{3}\right)^2 \cdot \frac{n}{4} - 1\frac{n}{12} = \frac{n}{36}, \\ |f'(\zeta)| &\geqq \frac{\sqrt{n}}{6}. \end{aligned} \tag{31}$$

We can show in a similar way that there exists a point $a < \eta \leqq a + \frac{2}{\sqrt{n}}$ for which $|f'(\eta)| \geqq \frac{1}{6}\sqrt{n}$ holds.

The proof of Turán's theorem is completed. The result will be used in § 4 in the following formulation: *Let the roots of polynomial $f(z)$ of degree n fall in the interval $(-1, +1)$ and let $f'(a) = 0$. Then there exist points ζ and η satisfying*

$$a - \frac{2}{\sqrt{n}} \leqq \zeta < a < \eta \leqq a + \frac{2}{\sqrt{n}};$$
$$|f'(\zeta)| \geqq \frac{1}{6}\sqrt{n} \cdot |f(a)| \text{ and } |f'(\eta)| \geqq \frac{1}{6}\sqrt{n} \cdot |f(a)|.$$

b) Turán's theorem is exact as to the order of magnitude, but it can be improved in the following way:

Theorem I. *Let the roots of the polynomial $f(z)$ of degree n lie in the interval $(-1, +1)$ and let $|f(a)| = 1$ for a point a, $-1 \leqq a \leqq +1$. Then there exists a point $-1 \leqq \zeta \leqq +1$ such that:*

in case of $n = 2, 3$,

$$|f'(\zeta)| \geqq \frac{n}{2}, \tag{32a}$$

in case of even $n \geqq 4$ $(n = 4, 6, 8, \dots)$

$$|f'(\zeta)| \geqq \frac{n}{\sqrt{n-1}}\left(1 - \frac{1}{n-1}\right)^{\frac{n-2}{2}} = \sqrt{\frac{n}{e}} + O\left(\frac{1}{n}\right), \tag{32b}$$

in case of odd $n \geqq 5$ $(n = 5, 7, 9, \dots)$

$$\begin{aligned} |f'(\zeta)| &\geqq \frac{n^2}{(n-1)\sqrt{n+1}}\left(1 - \frac{\sqrt{n+1}}{n-1}\right)^{\frac{n-3}{2}}\left(1 + \frac{1}{\sqrt{n+1}}\right)^{\frac{n-1}{2}} \\ &= \sqrt{\frac{n}{e}} + O\left(\frac{1}{n}\right). \end{aligned} \tag{32c}$$

Our theorem cannot be improved.

Proof. We search for the polynomial that satisfies the conditions of the theorem, and the maximum of its derivative is the smallest possible in the interval $(-1, +1)$. Let $f(z)$ be an arbitrary polynomial of degree n that satisfies the conditions. Let

$$H_n = \max_{-1 \leqq z \leqq 1} |f'(z)|. \tag{33}$$

We are looking for the value of

$$h_n = \min_f H_n. \tag{34}$$

Our theorem will obviously be verified if we prove that h_n is equal to the values given in (32a), (32b), (32c). Let us investigate now on which parameters the function H_n depends. Without restriction of generality we may assume that

$$f(a) = 1. \tag{35}$$

Let the roots of the equation $f(z) = 0$ be $z_1, z_2, \dots, z_n$, where

$$-1 \leqq z_1 \leqq z_2 \leqq \dots \leqq z_n \leqq +1. \tag{36}$$

Thus, the form of the polynomial is

$$f(z) = \frac{(z - z_1)(z - z_2)\dots(z - z_n)}{(a - z_1)(a - z_2)\dots(a - z_n)}. \tag{37}$$

So we are searching for the value of

$$h_n = \min H_n(a; z_1, z_2, \dots z_n),$$

where H_n is determined by (33). The existence of the minimum follows from Weierstrass' theorem.

First we shall prove that H_n can be made smaller, except for the cases when the absolute value of each root is 1, i.e., in the case of a polynomial yielding the minimum h_n, we have

$$|z_i| = 1 \quad (i = 1, 2, \dots, n). \tag{38}$$

Then we are going to choose out of finitely many polynomials the one which yields the minimum.

We may suppose that $|f(a)|$ is the maximum of $|f(z)|$ in the interval $(-1, 1)$, otherwise dividing by it the value of H_n would decrease. In a similar way as in proving Turán's theorem we separate the cases $|a| = 1$ and $|a| \neq 1$.

Let first $|a| = 1$. Then, on the basis of (22), $|f'(a)| \geqq \frac{n}{2}$. However, the polynomial $\left(\frac{1+z}{2}\right)^n$ satisfies the conditions, and for this the value of H_n

is $\frac{n}{2}$. So this case can be represented by the above polynomial which also satisfies (38).

Thus we have to deal with the case when there exists an $i \geqq 2$, for which

$$z_{i-1} < a < z_i.$$

In such a case the absolute maximum $|f(a)|$ is also a local maximum, so we have

$$f'(a) = 0. \tag{39}$$

Proving Turán's theorem we saw that in the vicinity of the point a $|f'(z)|$ takes some large values, while at the point a its value is zero by (39). So $|f'(z)|$ increases to the right and to the left of a in the interval $(-1, 1)$ up to a certain maximum. Let, for example, ζ $(-1 \leqq \zeta < 1)$ be the point of that maximum to the left of a. Let us change the polynomial $f(z)$ into $f_1(z)$ in such a way that instead of some inner root z_k we write z'_k, instead of a we write a' and divide by $f_1(a')$. Here a' is the point of maximum of $|f_1(z)|$ in the interval (z_{i-1}, z_i), and so it is explicitly determined. Namely, (39) holds: $f'_1(a') = 0$, and by Rolle's theorem it is easily seen that if all the roots of a polynomial are real, its derivative vanishes between any two consecutive of them once and only once. Let $|f'_1(\zeta')|$ be the maximum of $|f'_1(z)|$ in the interval $(-1, 1)$ to the left of a'. We shall prove that – provided there exists an inner root of $f(z)$ at all – there exists a k such that

$$|f'_1(\zeta')| < |f'(\zeta)|, \quad \text{if} \quad |z'_k - a| > |z_k - a|, \tag{40}$$

that is, if z_k is moved further from the point a. Let us assume for a moment that (40) is already verified. Let us remove all the roots from point a to the farthest possible left and right, i.e., let us take them into points -1 and $+1$, through which we get the polynomial $f_m(z)$. This polynomial possesses the property in (38) and also the conditions of the theorem, and on the basis of (40),

$$|f'_m(\zeta^{(m)})| < |f'(\zeta)|. \tag{41}$$

In addition, $|f'_m(\zeta^{(m)})|$ is the maximum of $|f'_m(z)|$ in the interval $(-1, a^{(m)})$, which is easy to see. When we write "to the right" instead of "to the left", the argumentation remains valid, we arrive again at a polynomial $f_m(z)$, but now the interval $(a^{(m)}, 1)$ plays a role. In such a way the maximum of $|f'_m(z)|$ is smaller in $(-1, 1)$ than the maximum of $|f'(z)|$, so with (40) we also verify (38). And we shall prove (40) if we show that when there are any inner roots, then there is one among them for which the inequality

$$\frac{d|f'(\zeta)|}{dz_k} \begin{cases} > 0 & \text{if } z_k < a \\ < 0 & \text{if } z_k > a \end{cases} \tag{42}$$

holds. For this purpose we calculate the value of $\frac{d|f'(\zeta)|}{dz}$ that exists, because $f'(\zeta) \neq 0$. If we change the root z_k of the polynomial, then also the locations of a and ζ change, so that

$$\frac{d|f'(\zeta)|}{dz_k} = \frac{\partial|f'(\zeta)|}{\partial z_k} + \frac{\partial|f'(\zeta)|}{\partial a} \cdot \frac{da}{dz_k} + \frac{\partial f'(\zeta)}{\partial \zeta} \cdot \frac{d\zeta}{dz_k}. \tag{43}$$

We shall show that the last two terms are zero. Indeed, $\frac{da}{dz_k}$ is a finite value, since with the roots of the polynomial the roots of its derivative changes smoothly (as a differentiable function). Furthermore,

$$f'(\zeta) = \frac{(\zeta - z_1)(\zeta - z_2)\dots(\zeta - z_n)}{(a - z_1)(a - z_2)\dots(a - z_n)} \sum_{i=1}^{n} \frac{1}{\zeta - z_i}, \tag{44}$$

and so, using equality (39) we obtain

$$\left|\frac{\partial|f'(\zeta)|}{\partial a}\right| = |f'(\zeta)| \cdot |f'(a)| = 0. \tag{45}$$

If $-1 < \zeta < 1$, $|f'(\zeta)|$ is a local maximum, so

$$f''(\zeta) = 0. \tag{46}$$

Similarly to the above, $\frac{d\zeta}{dz_k}$ exists and we have

$$\left|\frac{\partial|f'(\zeta)|}{\partial \zeta}\right| = |f''(\zeta)| = 0. \tag{47}$$

If $|\zeta| = 1$, then $d\zeta = 0$ by the identical arrangement of the roots of $f''(z)$, which is easy to show. If, on the other hand, $d\zeta = 0$, then the last term in (43) does not occur.

We have yet to calculate $\frac{\partial|f'(\zeta)|}{\partial z_k}$. Because in case of $f'(\zeta) \neq 0$ we get

$$\frac{1}{|f'(\zeta)|} \cdot \frac{\partial|f'(\zeta)|}{\partial z_k} = \frac{1}{f'(\zeta)} \cdot \frac{\partial f'(\zeta)}{\partial z_k},$$

so

$$\frac{d|f'(\zeta)|}{dz_k} = \frac{\partial|f'(\zeta)|}{\partial z_k} = |f'(\zeta)| \left[\frac{1}{a - z_k} - \frac{1}{\zeta - z_k} + \frac{f(\zeta)}{(\zeta - z_k)^2 f'(\zeta)}\right]. \tag{48}$$

Having done these calculations, (42) can be verified briefly. Let

$$h(z_k) = \frac{1}{a - z_k} - \frac{1}{\zeta - z_k} + \frac{f(\zeta)}{(\zeta - z_k)^2 f'(\zeta)}. \tag{49}$$

We have to show that (49) has the proper sign. Let $z_k < a$.

If we consider the expression (49) as a function of z_k we can see that it changes sign at $z_k = a$, because the term $\frac{1}{a-z_k}$ dominates there. The question is, where else $h(z_k)$ changes sign. Reformulating we get

$$h(z_k) = \frac{1}{(\zeta - z_k)^2}\left(\frac{f(\zeta)}{f'(\zeta)} + (\zeta - a)\frac{\zeta - z_k}{a - z_k}\right). \tag{50}$$

From this we can see that $h(z_k)$ changes sign apart from the point a only at one single point, denoted by x_0. If $x_0 > a$, then $h(z_k)$ does not change sign in interval $(-1, a)$, so it is everywhere positive, since it is positive near the point a at the end of the interval. In this case (40) is true for all roots $z_i < a$. On the other hand, if $x_0 < a$, then every root in the interval (x_0, a) satisfies (40). For all the rest of the roots $h(z_k) < 0$ holds. So, if there is no root of $f(z)$ in the interval (x_0, a), then the inequality $h(z_k) < 0$ would be true for all k, so

$$\begin{aligned} 0 > \sum_{k=1}^{n} h(z_k) &= \sum_{k=1}^{n}\left(\frac{1}{a-z_k} - \frac{1}{\zeta - z_k} + \frac{f(\zeta)}{f'(\zeta)}\cdot\frac{1}{(\zeta - z_k)^2}\right) \\ &= \frac{f'(a)}{f(a)} - \frac{f'(\zeta)}{f(\zeta)} + \frac{f(\zeta)}{f'(\zeta)}\cdot\frac{f'(\zeta)^2 - f(\zeta)f''(\zeta)}{f(\zeta)^2} = 0, \end{aligned}$$

provided $|\zeta| \neq 1$, by (39) and (46). This is a contradiction. So in case $|\zeta| \neq 1$ there exists some root complying with (40) and smaller than a. If $|\zeta| = 1$ we may obtain by linear substitution that $z_1 = -1$, $z_n = +1$ while the derivative decreases. So we may suppose that in this case $f(\zeta) = 0$, but $f'(\zeta) \neq 0$, so we have to prove that for $a > z_k$

$$\frac{1}{a - z_k} - \frac{1}{\zeta - z_k} > 0$$

which is evident in case of $\zeta = -1 < z_k$, and is easily verified in case of $\zeta = +1$, because $a - z_k < 1 - z_k$.

The proof is analogous when $z_k > a$. In this case we have to show that there exists a z_k, $1 > z_k > a$, for which $h(z_k) < 0$.

Therefore (40), and simultaneously (38), have been proved. Thus, on the basis of (38), (37) and (39) we conclude that the only polynomials that could provide the value h_n are:

$$f_k(z) = \frac{n^n}{2^n k^k (n-k)^{n-k}}(1-z)^k(1+z)^{n-k} \quad (k = 0, 1, 2, \ldots, n). \tag{51}$$

Polynomials $f_k(z)$ and $f_{n-k}(z)$ are reflections of each other, so it is sufficient to deal with the cases $k \leqq \frac{n}{2}$. Let us denote by $H_n(k)$ the value yielded by (51) through (33). Then $H_n(0) = \frac{n}{2}$, and $H_n(1) \geqq \frac{n}{2}$, so in case of $n = 2, 3$ we obtain $h_n = \frac{n}{2}$ which is just (32a).

In case of $k \geqq 2$, $|f_k'(z)|$ takes on two maxima in the interval $(-1,1)$, let us denote them by $H_n'(k)$ and $H_n''(k)$. But $H_n'(k) = H_n''(n-k)$ and we have

$$\begin{aligned} H_n'(k) &= \frac{n^2}{2\sqrt{(n-1)k(n-k)}}\left(1-\sqrt{\frac{n-k}{k(n-1)}}\right)^{k-1} \\ &\quad \times \left(1+\sqrt{\frac{k}{(n-k)(n-1)}}\right)^{n-k-1}. \end{aligned}$$

If n is even, we proceed as follows. Obviously

$$\begin{aligned} H_n(k) &\geqq \sqrt{H_n'(k)H_n''(k)} \\ &= \frac{n}{2}\sqrt{\frac{n^n}{(n-1)^{n-1}}\cdot\frac{1}{k(n-k)}\left(1-\frac{1}{k}\right)^{k-1}\left(1-\frac{1}{n-k}\right)^{n-k-1}} \end{aligned}$$

holds true. But

$$\frac{1}{k(n-k)} \geqq \frac{4}{n^2},$$

and if we examine the function $g(k) = \left(1-\frac{1}{k}\right)^{k-1}\left(1-\frac{1}{n-k}\right)^{n-k-1}$ in the interval $\left(2,\frac{n}{2}\right)$, we see that $g'(k) < 0$ there. Thus

$$\left(1-\frac{1}{k}\right)^{k-1}\left(1-\frac{1}{n-k}\right)^{n-k-1} \geqq \left(1-\frac{2}{n}\right)^{n-2}.$$

That is,

$$H_n(k) \geqq \sqrt{\frac{n^n(n-2)^{n-2}}{(n-1)^{n-1}n^{n-2}}} = \frac{n}{\sqrt{n-1}}\left(1-\frac{1}{n-1}\right)^{\frac{n-2}{2}} = H_n\left(\frac{n}{2}\right).$$

So (32b) is proved, provided

$$H_n(0) \geqq H_n\left(\frac{n}{2}\right)$$

holds for $n \geqq 4$, i.e., if

$$\frac{n}{2} \geqq \frac{n}{\sqrt{n-1}}\left(1-\frac{1}{n-1}\right)^{\frac{n-2}{2}},$$

which is clearly true.

The proof of (32c) can be carried out in a similar way, although on the basis of a much more complicated calculation. It can be verified that $\frac{dH_n'(k)}{dk} < 0$, if $k \geqq 2$, and that $H_n'\left(\frac{n-1}{2}\right) > H_n''\left(\frac{n-1}{2}\right)$ holds true. These verifications are not necessary to detail. The proof of Theorem I is completed.

§ 2

a) Let us consider the ellipse in the complex plane whose major axis is the interval $(-1,+1)$ and minor axis is $(-ai, ai)$ $(0 \leqq a \leqq 1)$. Let the collection of the inner and the boundary points of this ellipse be denoted by $\mathcal{E}$, and z_0 be a point on its boundary. Then the following theorem holds true.

Theorem II. *If the roots of a polynomial $f(z)$ of degree n lie in the closed domain $\mathcal{E}$, then we have*

$$|f'(z_0)| \geqq \frac{n \cdot a}{2\sqrt{1+a^2-|z_0|^2}}|f(z_0)| \geqq \frac{n \cdot a}{2}|f(z_0)|. \tag{52}$$

In the proof we may assume without loss of generality that the equality

$$f(z_0) = 1$$

holds true. We provide two proofs.

a) Let us examine the function

$$z(\zeta) = e^{i\varphi}\left(\frac{1+a}{2}\zeta + \frac{1-a}{2}\cdot\frac{1}{\zeta}\right), \tag{53}$$

which maps the unit circle $|\zeta| = 1$ onto the boundary of $\mathcal{E}$ and there we have

$$\left|\frac{dz}{d\zeta}\right| = \sqrt{1+a^2-|z|^2} \tag{54}$$

which can be proved by simple calculus. We are going to examine the function $g(\zeta) = f(z(\zeta))$ on the unit circle. We have to prove on the basis of (52) and (54) that

$$|g'(\zeta_0)| \geqq \frac{n \cdot a}{2},$$

where ζ_0 is the point on the unit circle corresponding to z_0. Let us choose φ in (53) so that $\zeta_0 = 1$. According to (53)

$$g(\zeta) = \frac{c_{-n}}{\zeta^n} + \cdots + \frac{c_{-1}}{\zeta} + c_0 + c_1\zeta + \cdots + c_n\zeta^n,$$

so $g(\zeta)\zeta^n$ is a polynomial of degree $2n$. Its roots are obtained pairwise from the second degree equation

$$z_k = e^{i\varphi}\left(\frac{1+a}{2}\zeta + \frac{1-a}{2}\cdot\frac{1}{\zeta}\right) \quad (k = 1, 2, \ldots, n), \tag{55}$$

where z_k is the root of the equation $f(z) = 0$. Let us denote the roots of $g(\zeta) = 0$ pairwise by ζ_k' and ζ_k'', then

$$\zeta_k' + \zeta_k'' = \frac{2z_k}{e^{i\varphi}(1+a)}; \quad \zeta_k'\zeta_k'' = \frac{1-a}{1+a}, \tag{56}$$

so

$$\left(g(\zeta)\zeta^n\right)'_{\zeta=1} = g'(1) + n = \sum_{k=1}^{n}\left(\frac{1}{1-\zeta_k'} + \frac{1}{1-\zeta_k''}\right) = n + a\sum_{k=1}^{n}\frac{1}{1-e^{-i\varphi}z_k},$$

that is,

$$|g'(1)| \geqq a \cdot \sum_{k=1}^{n} \Re\left(\frac{1}{1-e^{-i\varphi}z_k}\right) \geqq a \cdot \frac{n}{2}, \text{ [7]}$$

because $z_k \in \mathcal{E}$, and therefore $|z_k| \leqq 1$, the inequality

$$\Re\left(\frac{1}{1-e^{-i\varphi}z_k}\right) \geqq \frac{1}{2}$$

holds.

Thus, the proof of Theorem II is completed.

In case of $a = 1$, that is, in case of a circle Turán's result in (10) follows from our theorem. Turán's proof rather resembles the next, second proof.

b) In case of $f(z_0) = 1$ the estimate

$$|f'(z_0)| = \left|\sum_{k=1}^{n}\frac{1}{z_0 - z_k}\right| \geqq \frac{n \cdot a}{2} \cdot \frac{1}{\sqrt{1+a^2-|z_0|^2}}$$

is valid, if for some properly chosen φ

$$\Re\left(\frac{e^{i\varphi}}{z_0 - z_k}\right) \geqq \frac{a}{2} \cdot \frac{1}{\sqrt{1+a^2-|z_0|^2}} \tag{57}$$

holds for $k = 1, 2, \ldots, n$. Let φ be the angle enclosed by the normal direction drawn at point z_0 to $\mathcal{E}$ and by the positive direction of the real axis. In this case the left-hand side of (57) is not negative, so it decreases, if z_k moves in the direction of the vector $z_k - z_0$. So, we can restrict ourselves to the case when z_k is on the boundary of $\mathcal{E}$. Thus we may suppose that

$$z_0 = \cos\alpha + ia\sin\alpha, \quad z_k = \cos\beta + ia\sin\beta. \tag{58}$$

From the parametrical representation of the ellipse it follows that the directional tangent of its normal drawn at the point z_0 is

$$\operatorname{tg}\varphi = \frac{\operatorname{tg}\alpha}{a}. \tag{59}$$

Thus, based on the relation

$$\Re\left(\frac{a+ib}{c+id}\right) = \frac{ac+bd}{c^2+d^2}$$

[7] $\Re(z)$ stands for the real part of z.

we have by (58) and (59)

$$\begin{aligned}\Re\left(\frac{e^{i\varphi}}{z_0 - z_k}\right) &= \frac{\cos\varphi(\cos\alpha - \cos\beta) + a\sin\varphi(\sin\alpha - \sin\beta)}{(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2} \\ &= \frac{\cos\varphi}{2\cos\alpha\left(a^2 + (1-a^2)\sin^2\frac{\alpha+\beta}{2}\right)} \geqq \frac{\cos\varphi}{2\cos\alpha}.\end{aligned}$$

On the other hand, it follows from (59) that

$$\cos\varphi = \frac{a}{\sqrt{a^2 + \operatorname{tg}^2\alpha}}.$$

We obtain

$$\Re\left(\frac{e^{i\varphi}}{z_0 - z_k}\right) \geqq \frac{a}{2\sqrt{a^2\cos^2 a + \sin^2\alpha}} = \frac{a}{2\sqrt{1 + a^2 - |z_0|^2}},$$

and this is exactly (57).

c) Finally, we are going to prove that Theorem II is exact with respect to the order of magnitude in cases $a \geqq \frac{2}{7\sqrt{n}}$. If $a \leqq \frac{2}{7\sqrt{n}}$, we can show similarly to the proofs in § 1 that if $f(z) \in E(\mathcal{E})$, then

$$\frac{\max\limits_{z\in\mathcal{E}} |f'(z)|}{\max\limits_{z\in\mathcal{E}} |f(z)|} \geqq \frac{1}{7}\sqrt{n},$$

and this is also exact as to the order of magnitude. The last observation is contained in Theorem III.

Theorem III. *If the roots of the polynomial $f(z)$ of degree n lie in the interior of $\mathcal{E}$ and in a point z_0 of $\mathcal{E}$ $|f(z_0)| = 1$ holds, then there exists a point ζ on the boundary of $\mathcal{E}$ such that*

$$|f'(\zeta)| \geqq \max\left(\frac{n\cdot a}{2}, \frac{1}{7}\sqrt{n}\right) \geqq \frac{n\cdot a}{4} + \frac{1}{14}\sqrt{n}. \tag{60}$$

On the other hand, there exists a polynomial $f_1(z)$ for which the conditions above are fulfilled and

$$|f_1'(z)| \leqq \frac{3}{\sqrt{(1+a^2)^3}}(na + \sqrt{n}) \tag{61}$$

for the points of $\mathcal{E}$. This bound is smaller than 42 times the value given in (60).

Proof of (60) can be carried out on the basis of the previous part, applying the fact that a regular function attains the maximum of its absolute value on the boundary of the complex domain. Thus we may suppose that z_0 lies on the boundary of $\mathcal{E}$.

To verify (61) it is also sufficient to examine $|f'(z)|$ on the boundary of $\mathcal{E}$. Let first $n = 2m$ be even and $m \geqq 2$. If $n = 2, 3$, then also $f(z) = \left(\frac{1+z}{2}\right)^n$ can be used as f_1 and f_2 below.

Let

$$f_1(z) = \frac{(1-z^2)^m}{(1+a^2)^m}, \tag{62}$$

which satisfies the requirements. On the boundary of $\mathcal{E}$ we have

$$|f_1'(z)| < \frac{na}{1+a^2} + \sqrt{\frac{2}{e}}\frac{\sqrt{n}}{1+a^2} < \frac{na}{1+a^2} + \frac{\sqrt{n}}{1+a^2} \tag{63}$$

which states even more than (61).

In the case of odd n, $n = 2m+1$, let

$$f_2(z) = \frac{(1-z^2)^m(1+z)}{(1+a^2)^m\sqrt{1+a^2}} = \frac{1}{\sqrt{1+a^2}}\left(\frac{(1-z^2)^m}{(1+a^2)^m} + \frac{z(1-z^2)^m}{(1+a^2)^m}\right) \tag{64}$$

Then we obtain

$$|f_2'(z)| < \frac{3}{\sqrt{(1+a^2)^3}}\left(na + \sqrt{n}\right).$$

□

§ 3

Concerning Turán's problem we shall examine the question, for what kind of domains M the method of § 2 b) can be applied.

To this end let us look more closely at the argument in § 2 b). Let an arbitrary closed domain M be given, and also a polynomial $f(z)$ of degree n, whose roots lie in domain M. Let $|f(z_0)| = 1$ for a certain point z_0 on the boundary of this domain. Then we have

$$|f'(z_0)| = \left|\sum_{i=1}^{n} \frac{1}{z_i - z_0}\right|.$$

We may regard the complex numbers $\frac{1}{z_i - z_0}$ also as vectors. The most essential idea of § 2 b) is that there exists a direction such that the projections of the above vectors in this direction are unidirectional and exceed in length a positive constant c_1. If we verify this, we shall prove

$$|f'(z_0)| \geqq c_1 \cdot n, \tag{65}$$

where $c_1 > 0$ depends only on the domain M.

In order to have the direction in the above sense in each boundary point of M it is necessary that M be a convex domain or a convex curve. This direction is a normal line drawn at the point z_0 or in case of a vertex any line going through both the vertex and the domain. If this direction and $z_0 - z$ together enclose the angle α, then the projection of $\frac{1}{z-z_0}$ is $\frac{\cos\alpha}{|z-z_0|}$. We are looking for the value

$$\min_{z_0, z\in M} \frac{\cos\alpha}{|z-z_0|} = c_2$$

for the boundary points z_0. We have to prove that $c_2 > 0$. For convex domains $c_2 \geqq 0$ holds true. In case $c_2 = 0$ $\cos\alpha = 0$, i.e. z is a point of the tangent drawn at the point z_0. Let us assume that the boundary curve of M is nowhere straight. Then in case of $\cos\alpha = 0$ we must have $z \to z_0$ on the boundary curve of M. If ϱ is the radius of the circle of curvature in z_0, then by the definition of ϱ,

$$\frac{|z_0 - z|}{\frac{\pi}{2} - \alpha} \to \varrho, \quad \text{hence} \quad \frac{\cos\alpha}{|z_0 - z|} \to \frac{1}{\varrho}.$$

So, if on the boundary curve of the convex domain M apart from the vertices there exists a curvature radius and it is bounded, then (65) holds true.

We may relax the above conditions. Let us suppose that (65) is not valid for the convex domain M, i.e., $E(M)$ contains a polynomial of degree n for which, in case of $z \in M$,

$$|f'(z)| \leqq c \cdot n,$$

where $c > 0$ is arbitrary, $n > N(c)$.

Let us assume that $|f(z)|$ takes on its maximum in the domain M at the point z_0. If the curvature radius exists at z_0 and is finite, or z_0 is a vertex such that the difference of the inclination angles of the tangents is smaller than π, then according to the previous arguments there exists a number c' independent of n such that

$$|f'(z_0)| \geqq c' \cdot n.$$

Let the boundary of M be straight at point z_0, and let

$$f(z) = \alpha \prod_{i=1}^{n} (z - z_i).$$

Then, because of the maximality of $|f(z_0)|$, we have

$$\prod_{i=1}^{n} |z_0 - z_i| \geqq \prod_{i=1}^{n} |z - z_i| \tag{66}$$

for $z \in M$.

We assume about the domain M that it has a point z for which $|z-z_0| > 1$ holds, and on the other hand, for each of its points z, $|z-z_0| < 1+\delta_1$ (where we specify $\delta_1 > 0$ later). This can be assumed without loss of generality since if (65) is valid for a domain, then with a suitably changed c it is also valid for similar domains, which can be verified by a linear transformation.

Let a be larger than the length of the line segment bounding M at the point z_0. If (65) is not valid for M, then at least $\frac{n}{\delta_2}$ roots fall into the interior of the circle $|z_0 - z| \leqq a$, otherwise for $n\left(1-\frac{1}{\delta_2}\right)$ roots we have

$$\frac{\cos\alpha}{|z-z_0|} > c'',$$

as discussed above, and so

$$|f'(z_0)| \geqq c''\left(1-\frac{1}{\delta_2}\right)n$$

follows. The value of $\delta_2 > 1$ will also be specified later.

For the $\frac{n}{\delta_2}$ roots the inequality $|z_0 - z_i| < a$ is valid, while for the rest of the zeros we have $|z_0 - z_i| < 1+\delta_1$. Thus

$$\prod_{i=1}^{n} |z_0 - z_i| < a^{\frac{n}{\delta_2}}(1+\delta_1)^{\frac{n}{\delta_2}(\delta_2-1)}. \tag{67}$$

The next theorem due to Chebyshev's[8] is well-known.

Let $g(z)$ be a polynomial of degree n, in which the coefficent of z^n is 1. *Then the maximum of the absolute value of g in the interval $-1 \leqq z \leqq 1$ is at least $\frac{1}{2^{n-1}}$, i.e.,*

$$\max_{-1\leqq z\leqq 1} |g(z)| \geqq \frac{1}{2^{n-1}}.$$

Equality holds only for the Chebyshev polynomial.

Among the radii of the circle $|z_0 - z| = 1$ there is a radius which lies in M, so it contains a point z different from z_0 and such that

$$\prod_{i=1}^{n} |z - z_i| \geqq 4^{1-n}. \tag{68}$$

By (66), (67) and (68) we obtain

$$a^{\frac{n}{\delta_2}}(1+\delta_1)^{\frac{n}{\delta_2}(\delta_2-1)} \geqq 4^{1-n},$$

[8] G. Faber: *Über Tschebyscheffsche Polynome.* Journal für die reine und angew. Math. **150** (1919), 79–106.

or

$$a \geqq \frac{\sqrt[n]{4^{\delta_2}}}{4^{\delta_2}(1+\delta_1)^{\delta_2-1}} = A. \tag{69}$$

If $a > 4^{-\left(1-\frac{1}{n}\right)}$, then $\delta_1 > 0$ and $\delta_2 > 1$ can be chosen so that $a < A$. That contradicts (69). So, summarizing the above, as we have used here only the assumption that there are intervals in the set longer than 1, the following theorem holds.

Theorem IV. *Let a bounded, convex set M of points be given, bounded by closed Jordan arcs, which is identical*

either with a curve whose curvature exists and is nowhere zero,

or with a domain whose boundary is nowhere a piece of line whose length was one quarter of the diameter, or even longer.

Let the roots of the polynomial $f(z)$ of degree n lie in the domain M and let $|f(z)|$ take on its maximum there in the point z_0.

Then there exists a constant $c > 0$ depending only on M such that

$$|f'(z_0)| > c \cdot n \cdot |f(z_0)|.$$

This result can be made more accurate by Faber's theorem[8]. If $z = \psi(x)$ takes the circle $|x| = r$ onto the boundary curve of M, and also $x = \infty$ into the origin, without scaling then there can exist a piece of line whose length is $a < \frac{1}{r}$.

§ 4

a) Pál Erdős studied the following problem.

Let $z_1, z_2, \ldots, z_n$ be the roots of the polynomial $f(z)$ of degree n, and let

$$-1 \leqq z_1 \leqq z_2 \leqq \cdots \leqq z_n \leqq 1.$$

Assume that the polynomial is convex (concave) from below in the interval (z_{i-1}, z_i). What can we claim about the distance between two consecutive roots?

Erdős, and independently also Turán, showed that under the above conditions

$$z_i - z_{i-1} \leqq \frac{16}{\sqrt{n}} \tag{70}$$

holds true.

Our proof is as follows. Let a be the root of equation $f'(z) = 0$ falling in the interval (z_{i-1}, z_i). Then, by Turán's theorem (see § 1 a)), there exists a point ζ such that

$$0 < a - \zeta \leqq \frac{2}{\sqrt{n}} \text{ and } |f(\zeta)| \geqq \frac{1}{6}\sqrt{n}|f(a)|. \tag{71}$$

Besides, convexity means that $|f'(z)|$ monotonically decreases between z_{i-1} and a, so if $z_{i-1} \leqq z \leqq \zeta$, then we have

$$|f'(z)| \geqq \frac{1}{6}\sqrt{n} \cdot |f(a)|.$$

But

$$|f(\zeta) - f(z_{i-1})| = |f(\zeta)| = \left| \int_{z_{i-1}}^{\zeta} f'(z)dz \right| = \int_{z_{i-1}}^{\zeta} |f'(z)|dz$$

$$\geqq (\zeta - z_{i-1}) \cdot \frac{1}{6}\sqrt{n} \cdot |f(a)|,$$

and, on the other hand, $|f(\zeta)| \leqq |f(a)|$. So, the estimate

$$\zeta - z_{i-1} \leqq \frac{6}{\sqrt{n}} \tag{72}$$

follows. According to (71) and (72) we have

$$a - z_{i-1} \leqq \frac{8}{\sqrt{n}},$$

and similarly $z_i - a \leqq \frac{8}{\sqrt{n}}$. This yields

$$z_i - z_{i-1} \leqq \frac{16}{\sqrt{n}}.$$

□

b) The previous theorem can be further improved. A precise result in this direction is the following.

Theorem V. *Under the previous conditions the following hold true:*
a) *In case of* n *even,*

$$z_i - z_{i-1} \leqq \frac{2}{\sqrt{2n-3}}\ ; \tag{73a}$$

b) *In case of* $n \geqq 3$ *odd,*

$$z_i - z_{i-1} \leqq \frac{2}{\sqrt{2n-3}} \cdot \frac{\sqrt{n^2-2n}}{n-1}. \tag{73b}$$

These estimates cannot be improved.

Before turning to our proof we verify a lemma.

Lemma. *Let $z_1 \leqq z_2 \leqq \cdots \leqq z_n$ be the roots of the equation $f(z) = 0$ of degree n, and $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-2}$ be the roots of $f''(z) = 0$. The latter are obviously functions of $z_1, z_2, \ldots, z_n$, i.e.,*

$$x_i = x_i(z_1, z_2, \ldots, z_n). \tag{74}$$

We shall show that

$$\frac{n-2}{n} \geqq \frac{dx_i}{dz_k} \geqq 0 \quad \begin{pmatrix} i = 1, 2, \ldots, n-2 \\ k = 1, 2, \ldots, n \end{pmatrix} \tag{75}$$

and

$$\frac{dx_i}{dz_k} = \frac{2}{(x_i - z_k)^2 f'''(x_i)} \left(\frac{f(x_i)}{x_i - z_k} - f'(x_i) \right). \tag{76}$$

Proof. First we prove (75). If we show that

$$\frac{dx_i}{dz_k} \geqq 0, \tag{77}$$

then (75) would follow because the relation between the coefficients and the roots of polynomials implies

$$z_1 + z_2 + \cdots + z_n = \frac{n}{n-2}(x_1 + x_2 + \cdots + x_{n-2}),$$

from which

$$\sum_{i=1}^{n} \frac{dx_i}{dz_k} = \frac{n-2}{n} \tag{78}$$

follows. On the left-hand side of (78) all summands are non-negative (by our assumption), so any term is smaller than the sum.

The relation (77) states that if we change a root of the polynomial, then any of the roots of the second derivative moves also in that direction.

Obviously it suffices to verify this claim showing that it holds if we take the first derivative instead of the second one. The roots of the first derivative are provided by the equation

$$\sum_{i=1}^{n} \frac{1}{z - z_i} = 0.$$

The point z_i changes in this equation. But $\frac{1}{z-z_i}$, as a function of the variable z_i, is monotonically increasing, and this already implies that the root of

$\sum_{i=1}^{n} \frac{1}{z-z_i} = 0$ moves in the direction of z_i's motion. Of course, this argument is only valid when all roots are real.

The proof of (76) can be done by a simple calculation.

Now we turn to the proof of our theorem. We are looking for the polynomial of degree n which satisfies the conditions and has the longest possible interval (z_{i-1}, z_i) where the polynomial is convex from below, i.e., $f''(z) \geqq 0$.

First we shall prove that no root of the polynomial can fall in the interval where it is convex from below. Let namely (z_i, z_k) be such an interval. Then we claim that $f''(z) > 0$ if $z_i < z < z_k$. That is to say, if $f''(\zeta) = 0$ occurs at a certain point, then it is at least a double root. But each root of $f(z)$ is real, so between any two roots of $f'(z)$ there is a root of $f(z)$, and at a k-fold root of $f'(z)$ the polynomial $f(z)$ has a $(k+1)$-fold root. Thus, if ζ is the smallest root of $f''(z)$ inside of interval (z_i, z_k), then $f(\zeta) = 0$ and $f'(\zeta) = 0$ will follow. But since $f(z_i) = f(\zeta) = 0$, by Rolle's theorem we conclude that $f'(z)$ vanishes at some point η, $z_i < \eta < \zeta$, and $f'(\eta) = f'(\zeta) = 0$ implies $f''(\xi) = 0$, where $\eta < \xi < \zeta$. But this contradicts the fact that $f''(z)$ has no roots in the interval (z_i, ζ). Hence, no roots of either $f''(z)$ or $f(z)$ fall between z_i and z_k, and the points z_i and z_k themselves cannot be multiple roots, because again by Rolle's theorem $f''(z)$ would also have a root between z_i and z_k. Hence i and k are two consecutive numbers: let us denote them by $i-1$ and i.

Assume that $f(z)$ is convex from below in the interval (z_{i-1}, z_i), and let us examine the situations in which this interval can be increased. Obviously, this will take place if we can alter the roots of $f(z)$ in such a way that $z_i - z_{i-1}$ increases and $f''(z) \neq 0$, if $z_{i-1} < z < z_i$. Firstly, we can show that such an alteration exists, except for the case when all roots but z_{i-1}, z_i fall into the end-points $+1$ and -1, furthermore $f''(z_{i-1}) = 0$ holds (with the exception $i = 2$, when $z_1 = -1$) and $f''(z_i) = 0$ (with the exception $i = n$, when $z_n = 1$).

Let us assume first that $i = 2$. By a linear transformation we can get another polynomial for which $z_1 = -1$, $z_n = 1$ hold true, and meanwhile (z_1, z_2) does not shrink, and also the convexity is preserved, so $z_2 \leqq x_1$. Let us move all the roots $z_3, z_4, \ldots, z_n$ into the point $+1$. By (77) x_1 will increase, the convexity remains, $z_2 \leqq x_1$ will be valid. Now we increase z_2 until $z_2 = x_1$, through which (z_1, z_2) obviously increases. Our polynomial thus already fulfils the aim posed in the previous paragraph. In case of $i = n$ the procedure is analogous.

Now let us suppose that $3 \leqq i \leqq n-1$. First we shall prove that except for the case

$$z_{i-1} = x_{i-2}, \quad z_i = x_{i-1} \tag{79}$$

the interval can be expanded. No root falls in the interval, so we may assume by Rolle's theorem that

$$x_{i-2} \leqq z_{i-1} < z_i \leqq x_{i-1}.$$

If on both sides the inequality holds, then we can decrease z_{i-1} until somewhere equality occurs. If, however, for instance

$$x_{i-2} = x_{i-1} < z_i < x_{i-1} \tag{80}$$

holds, then we decrease the root z_{i+1}, then by (77) x_{i-2} will also decrease, we arrive at the previous case and the interval can be increased.

Let us assume that there are at least two different points inside $(-1, +1)$, besides z_{i-1} and z_i, where $f(z)$ vanishes. Then we can achieve the situation in (80) by pushing out these roots properly, i.e., we can increase the interval (z_{i-1}, z_i). Let us denote by a and b two such roots of $f(z)$. We can move these roots in both directions. Let us perform with the root a a c-times larger move than with b (we shall specify c later). Then applying (76) we obtain

$$\frac{dx_{i-2}}{da} = -\frac{2f'(x_{i-2})}{f'''(x_{i-2})}\left\{\frac{c}{(x_{i-2}-a)^2} + \frac{1}{(x_{i-2}-b)^2}\right\},$$
$$\frac{dx_{i-1}}{da} = -\frac{2f'(x_{i-1})}{f'''(x_{i-1})}\left\{\frac{c}{(x_{i-1}-a)^2} + \frac{1}{(x_{i-1}-b)^2}\right\}.$$

The value of c can obviously be chosen so that $\frac{dx_{i-2}}{da}$ and $\frac{dx_{i-1}}{da}$ have different signs. Indeed, by virtue of $\frac{dx_{i-2}}{dz_k} > 0$ we have

$$-\frac{2f'(x_{i-2})}{f'''(x_{i-2})} > 0,$$

and in the same way

$$-\frac{2f'(x_{i-1})}{f'''(x_{i-1})} > 0.$$

So, if c falls into the interval

$$\left(-\frac{(x_{i-1}-a)^2}{(x_{i-1}-b)^2}, -\frac{(x_{i-2}-a)^2}{(x_{i-2}-b)^2}\right),$$

then the two expressions have opposite signs. This interval is not of zero length. Moving a into the proper direction we would arrive to the case of (80). That is, the interval (z_{i-1}, z_i) could be increased.

We may suppose that the polynomial vanishes at three points in the interior of the interval $(-1, +1)$. Without loss of generality we may assume that these points are $z_{i-1} < z_i < z_{i+1}$. If $i + 1 = n$, we can again achieve $z_n = +1$ by a linear transformation, and meanwhile getting longer interval (z_{i-1}, z_i).

Part of our aim is thus achieved. In case of $i \leqq n-2$ we proceed analogously as before.

So, we proved that in connection with our task it is sufficient to examine polynomials of the form

(81)
$$\left.\begin{aligned} &f'(z) = (z+1)(z-a)(z-1)^{n-2}, \\ \text{where} \quad &f''(a) = 0; \quad -1 \leqq a \leqq 1 \end{aligned}\right\}$$

and

(82)
$$\left.\begin{aligned} &f(z) = (z+1)^{k-1}(z-a)(z-b)(z+1)^{n-k-1}; \\ &\qquad (k = 2, 3, \dots, n-2), \\ \text{where} \quad &f''(a) = f''(b) = 0; \quad -1 < a < b < 1. \end{aligned}\right\}$$

In (81) the length of the convex interval is:

$$1 + a = \frac{2}{n-1},$$

and in (82) it is

$$(b-a)^2 = \frac{4}{2n-3} - \frac{4(2k-n)^2}{(n-1)^2(2n-3)}.$$

This latter is then the largest, when $(2k-n)^2$ is the smallest. Hence the proof of the theorem can easily be finished.

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