On the Theory of Orthogonal Function Systems. *) **)

(First communication.)

By

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*) This paper is, except for insignificant modifications, a reprint of my inaugural dissertation from Göttingen, which appeared in July 1909.

**) Translated by Georg Zimmermann. Translator's Note: As nearly a century has elapsed since its publication, the original text sounds somewhat archaic to the modern ear. The translator has attempted to preserve the author's sound and style as much as possible. Some minor typographical errors have been corrected.

Introduction.

In the theory of series expansions of real functions, the so-called *orthogonal function systems* play a major rôle. By this, we mean a system of infinitely many functions $\varphi_1(s)$, $\varphi_2(s)$, $\cdots$ which have, with respect to an arbitrary, measurable set M of points, the *orthogonality property*

$$\int\limits_{(M)} \varphi_p(s)\,\varphi_q(s)\,ds = 0 \qquad (p \neq q,\ p, q = 1, 2, \cdots),$$

$$\int\limits_{(M)} \big(\varphi_p(s)\big)^2\,ds = 1 \qquad (p = 1, 2, \cdots),$$

where the integrals are taken in the Lebesgue sense; if they furthermore satisfy the so-called *completeness relation*

$$\int\limits_{(M)} \big(u(s)\big)^2\,ds = \Big\{ \int\limits_{(M)} u(s)\,\varphi_1(s)\,ds \Big\}^2 + \Big\{ \int\limits_{(M)} u(s)\,\varphi_2(s)\,ds \Big\}^2 + \cdots$$

for all functions $u(s)$ which together with their squares are integrable over the set M, then, following Hilbert, we denote the system a *complete orthogonal function system*, or, for short, a *complete orthogonal system* for the measure space M.

The formal infinite series

$$\varphi_1(s) \int\limits_{(M)} f(t)\,\varphi_1(t)\,dt + \varphi_2(s) \int\limits_{(M)} f(t)\,\varphi_2(t)\,dt + \cdots$$

is denoted the *Fourier series* of $f(s)$ with respect to the orthogonal function system $\varphi_1(s)$, $\varphi_2(s)$, $\cdots$.

The most simple orthogonal system is the system of trigonometric functions (for the interval $0 \leq s \leq 2\pi$)

$$\frac{1}{\sqrt{2\pi}}\,,\ \frac{1}{\sqrt{\pi}}\cos s\,,\ \frac{1}{\sqrt{\pi}}\sin s\,,\ \cdots,\ \frac{1}{\sqrt{\pi}}\cos ns\,,\ \frac{1}{\sqrt{\pi}}\sin ns\,,\ \cdots.$$

A large and interesting class of orthogonal function systems stems from the so-called *eigenvalue problem* for self-adjoint differential equations. This problem consists of determining those values for the parameter λ, for which the differential equation

$$\frac{d}{dx}\Big(p(x)\,\frac{du}{dx}\Big) + q(x)\,u + \lambda\,u = 0$$

has a solution which at two points $x = \alpha$ and $x = \beta$, say, satisfies homogeneous boundary conditions, like, *e.g.*,

$$u(\alpha) = 0 \quad \text{and} \quad u(\beta) = 0\,,$$

or

$$\frac{du}{dx} - h\,u = 0 \quad \text{for} \quad x = \alpha\,, \quad \text{and} \quad \frac{du}{dx} + H\,u = 0 \quad \text{for} \quad x = \beta\,.$$

It can be shown that if the functions $p(x)$ and $q(x)$ satisfy certain continuity conditions, then there always exist countably many such parameter values, and that the associated solutions form a complete orthogonal function system for the interval under consideration. Of particular importance is the so-called *regular* case, where the function $p(x)$ does not vanish on the interval (including the endpoints). The functions obtained in this manner are denoted a *Sturm-Liouville function system.* The case where the function $p(x)$ vanishes at one or both ends of the interval $[\alpha, \beta]$ is no less important; the *spherical harmonics* and the *Bessel functions* satisfy such a differential equation, as we know.

If we consider the by now classical theory of trigonometric series, we find that the results in this theory can be classified in four groups. First, we should name the

theory of convergence, whose duty it is to determine sufficient conditions on a function ensuring convergence of its trigonometric series. Right next to these studies, there is the

theory of divergence, who complements the former in many ways; it draws the lines showing how far the theory of convergence can reach at most. The most important result of this theory is the theorem by Du Bois-Reymond, predicting the existence of a continuous function whose trigonometric series does not converge. This makes a

theory of summation necessary, who is called upon to help out in the cases of divergence. Indeed, various summations methods are known with the aid of which it is possible to "sum" the trigonometric series of all continuous functions. The modern theory of summation of the trigonometric series was founded by L. Fejér; later on, various results by Poisson and Riemann were interpreted as summation methods by various authors. The last and most difficult problems are encountered in the

theory of uniqueness, which by its main problem — under which circumstances is a convergent trigonometric series the Fourier series of the represented function — forms the key stone of the whole theory. By the famous papers of Riemann, Cantor, and Du Bois-Reymond we are already able to answer also these questions.

As to the theory of the orthogonal functions originating from second order differential equations, which are closely related to the trigonometric functions, only the theory of convergence has been studied up to now. By a series of papers *) it has been proven that the conditions stated in the theory of trigonometric series, here also are sufficient to ensure convergence of the series. Only the theory of spherical harmonics has been pursued beyond these results in a recently published paper by Mr. Fejér **), in which the author discusses the summation theory of this function system.

In the paper in hand, we are dealing with the *theory of divergence and the theory of summation of orthogonal function systems.*

In Chapter I, the theory of divergence is discussed; § 1 presents a general sufficient condition, which in many cases enables us to construct for a given orthogonal function system a continuous function whose Fourier series with respect to this orthogonal system does not converge. In § 2 and § 3 this theorem is applied to the theory of Sturm-Liouville functions and to spherical harmonics in order to

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*) Of the many papers addressing this question I only name the more recent studies by Stekloff, Zaremba, Kneser, Hilbert, and Hobson.

**) Math. Annalen Vol. 67, p. 76.

construct a continuous function that can not be expanded with respect to these function systems.

Chapter II is dedicated to the theory of summation; in § 1, we prove a general lemma, which enables us to establish the converse of the theorem proved in § 1 of the first chapter. § 2 and § 3 present the application of this lemma to Sturm-Liouville series. This yields the result (which is a generalization of a theorem proved for trigonometric series by L. Fejér) that if a continuous function — which possibly has to satisfy certain boundary conditions — is expanded into a Sturm-Liouville series, and from the partial sums s_n of this series, the arithmetic means

$$s_1 \,,\ \frac{s_1+s_2}{2} \,,\ \frac{s_1+s_2+s_3}{3} \,,\ \dots \,,\ \frac{s_1+s_2+\cdots+s_n}{n} \,,\ \dots$$

are formed, then the sequence of functions thus defined converges uniformly to the given function. § 4 presents a general criterion which allows to decide whether a given summation method has the property that by its means, the Fourier series with respect to a given orthogonal system of all functions in the "range" of this system are summable.

The investigations in Chapter I suggest the question: does there exist at all an orthogonal function system with the property that *every continuous function can be expanded in the Fourier manner into a uniformly convergent series*, according to the functions of this system? In Chapter III, we shall encounter a *whole class* of orthogonal systems having this property. But these functions systems are also of interest from a different point of view, namely, because of a number of properties distinguishing this class. These properties point to the fact that in certain problems, where the orthogonal systems are used as an auxiliary means only, it will be advisable to employ just these special systems, whereby in many cases we gain a simpler presentation of the proof. In many cases still the nature of the very problem requires the application of such a special function system, without which the solution of the problem does not seem possible.

Chapter I.

Divergent Series.

If the functions

$$\varphi_1(s) \,,\ \varphi_2(s) \,,\ \cdots \,,\ \varphi_n(s) \,,\ \cdots$$

defined on the interval $[\alpha, \beta]$ form a complete orthogonal function system, then the formal series

$$\varphi_1(s) \int_\alpha^\beta f(t)\, \varphi_1(t)\, dt + \varphi_2(s) \int_\alpha^\beta f(t)\, \varphi_2(t)\, dt + \cdots$$

shall be denoted the *Fourier series* of the function $f(s)$ with respect to this orthogonal system. Terminating this infinite series at the n^{th} term, we obtain the finite sum

$$\varphi_1(s)\int\limits_\alpha^\beta f(t)\,\varphi_1(t)\,dt+\cdots+\varphi_n(s)\int\limits_\alpha^\beta f(t)\,\varphi_n(t)\,dt\,,$$

to which we want to refer as $[f(s)]_n$ from now on. Writing

$$K_n(s,t)=\varphi_1(s)\,\varphi_1(t)+\varphi_2(s)\,\varphi_2(t)+\cdots+\varphi_n(s)\,\varphi_n(t)$$

for short yields

$$[f(s)]_n=\int\limits_\alpha^\beta K_n(s,t)\,f(t)\,dt\,.$$

§ 1.

A General Criterion.

We base our investigations on an arbitrary orthogonal function system for the interval $[\alpha,\beta]$:

$$\varphi_1(s)\,,\;\varphi_2(s)\,,\;\cdots\,.$$

We denote by a an arbitrary point of this interval and consider the infinitely many numbers

$$\omega_n=\int\limits_\alpha^\beta \big|K_n(a,t)\big|\,dt\,;$$

if the numbers ω_n thus defined do not all lie below a finite bound, i.e.*, if from the sequence*

$$\omega_1\,,\;\omega_2\,,\;\omega_3\,,\;\cdots\,,$$

we can take a subsequence

$$\omega_{\nu_1}\le\omega_{\nu_2}\le\omega_{\nu_3}\le\cdots$$

whose elements grow beyond all bounds, then it is always possible to construct a continuous function whose Fourier series with respect to the orthogonal system at hand diverges at the point $s=a$.

The construction of this function $F(s)$ takes place in three steps.

1) *First*, we construct the both integrable and square integrable functions

$$v_{\nu_1}(s)\,,\;v_{\nu_2}(s)\,,\;v_{\nu_3}(s)\,,\;\cdots\,,$$

defined by the equation

$$v_{\nu_p}(s)=\text{sign of }K_{\nu_p}(a,s)\,;$$

i.e.,

$$\begin{aligned} v_{\nu_p}(s) &= 1, \quad \text{if } K_{\nu_p}(a,s)>0\\ &=-1, \quad ,, \qquad\qquad <0\\ &= 0, \quad ,, \qquad\qquad =0\,; \end{aligned}$$

thus it always holds that

$$v_{\nu_p}(t)\,K_{\nu_p}(a,t)=\left|K_{\nu_p}(a,t)\right|,$$

and consequently we have, in our notation,

$$[v_{\nu_p}(a)]_{\nu_p}=\int_\alpha^\beta \left|K_{\nu_p}(a,t)\right|dt=\omega_{\nu_p}\,.$$

The functions $v_{\nu_p}(s)$, who have absolute value ≤ 1 everywhere, thus have the property that the ${\nu_p}^{\text{th}}$ partial sum of their Fourier series at the point $s=a$ has value ω_{ν_p}.

2) Next, we construct a sequence of continuous functions

$$f_{\nu_1}(s)\,,\ f_{\nu_2}(s)\,,\ f_{\nu_3}(s)\,,\ \cdots$$

of absolute value less than 1 and having the property that

$$\int_\alpha^\beta \big(v_{\nu_p}(s)-f_{\nu_p}(s)\big)^2\,ds<\delta_p \qquad (p=1,2,3,\cdots),$$

where δ_p stands for an arbitrarily small positive quantity. *)

Forming the ${\nu_p}^{\text{th}}$ partial sum of the expansion of $f_{\nu_p}(s)$ yields

$$\begin{aligned}\left|[f_{\nu_p}(a)]_{\nu_p}\right| &= \left|\int_\alpha^\beta K_{\nu_p}(a,t)\,f_{\nu_p}(t)\,dt\right| \\ &= \left|\int_\alpha^\beta K_{\nu_p}(a,t)\,v_{\nu_p}(t)\,dt-\int_\alpha^\beta K_{\nu_p}(a,t)\,\big(v_{\nu_p}(t)-f_{\nu_p}(t)\big)\,dt\right| \\ &\geq \omega_{\nu_p}-\left|\int_\alpha^\beta K_{\nu_p}(a,t)\,\big(v_{\nu_p}(t)-f_{\nu_p}(t)\big)\,dt\right| \\ &\geq \omega_{\nu_p}-\sqrt{\int_\alpha^\beta \big(K_{\nu_p}(a,t)\big)^2\,dt\int_\alpha^\beta \big(v_{\nu_p}(t)-f_{\nu_p}(t)\big)^2\,dt}\,.\end{aligned}$$

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*) The construction of these functions does not pose any difficulties at all. A possible procedure is the following: let us assume for simplicity that $[0,2\pi]$ is the interval under consideration, which after all is no substantial restriction; we let

$$f_{\nu_p}(r,s)=\int_0^{2\pi}\frac{1-r^2}{1-2\,r\,\cos(s-t)+r^2}\,v_{\nu_p}(t)\,dt \qquad (0<r<1).$$

In the theory of trigonometric series it is shown that the continuous functions f_{ν_p} remain smaller than the maximum of $\left|v_{\nu_p}(s)\right|$ in absolute value, and that

$$\underset{r=1}{L}\int_0^{2\pi}\left[f_{\nu_p}(r,s)-v_{\nu_p}(s)\right]^2 ds=0\,.$$

Thus it is possible to determine r such that $f_{\nu_p}(r,s)$ satisfies all conditions posed.

If we now choose the quantity δ_p such that

$$\sqrt{\delta_p \int\limits_{\alpha}^{\beta} \big(K_{\nu_p}(a,t)\big)^2\, dt} < \frac{\omega_{\nu_p}}{2}\,,$$

then

$$\big|[f_{\nu_p}(a)]_{\nu_p}\big| > \frac{\omega_{\nu_p}}{2}\,.$$

In other words, the continuous functions $f_{\nu_p}(s)$, who remain less than 1 in absolute value, have the property that the ${\nu_p}^{\text{th}}$ partial sum of their Fourier series at the point $s = a$ turns out to be larger than $\frac{\omega_{\nu_p}}{2}$.

3) We now reach the sought-after function $F(s)$ by the following consideration: the ${\nu_1}^{\text{th}}$ partial sum of the Fourier expansion of the continuous function

$$F'(s) = f_{\nu_1}(s)$$

at the point $s = a$ is larger than $\frac{\omega_{\nu_1}}{2}$ in absolute value. If this series does not diverge at this point, we can determine a number G' such that all partial sums of the Fourier series of $F'(s)$ for $s = a$ are less than G', *i.e.*, that

$$\big|[F'(a)]_n\big| < G' \qquad\qquad (n = 1, 2, 3, \cdots).$$

We now pick out from the sequence of indices

$$\nu' = \nu_1\,,\ \nu_2\,,\ \nu_3\,,\ \cdots \tag{1}$$

an index which we want to call ν'', say, in such a manner that

$$\omega_{\nu''} > 6 \cdot 4(G' + 1)\,,$$

and then form with the associated function $f_{\nu''}(s)$ the continuous function

$$F''(s) = f_{\nu'}(s) + \frac{1}{4}\, f_{\nu''}(s)\,.$$

If the Fourier series of this continuous function $F''(s)$ is not divergent, we can determine a number G'' such that for each n, we have

$$\big|[F''(a)]_n\big| < G''\,.$$

Then we determine in the index sequence (1) an index ν''' in such a manner that the associated $\omega_{\nu'''}$ satisfies

$$\omega_{\nu'''} > 6 \cdot 4^2(G'' + 2)\,,$$

and form the function

$$F'''(s) = f_{\nu'}(s) + \frac{1}{4}\, f_{\nu''}(s) + \frac{1}{4^2}\, f_{\nu'''}(s)\,.$$

In this manner we keep proceeding: if the Fourier series of the continuous function

$$F^{(q-1)}(s) = f_{\nu'}(s) + \frac{1}{4}\, f_{\nu''}(s) + \cdots + \frac{1}{4^{q-2}}\, f_{\nu^{(q-1)}}(s)$$

does not diverge at the point $s = a$, then we determine $G^{(q-1)}$ such that for each n,

$$\big|[F^{(q-1)}(a)]_n\big| < G^{(q-1)} \tag{2}$$

and then pick out from the sequence (1) an index $\nu^{(q)}$ in such a manner that

$$\omega_{\nu^{(q)}} > 6 \cdot 4^{q-1}(G^{(q-1)} + q - 1)\,; \tag{3}$$

the possibility of this choice is guaranteed by the assumption that the ω_{ν_p} grow beyond all bounds.

Now I claim that the *infinite series*

$$F(s) = f_{\nu'}(s) + \frac{1}{4}\, f_{\nu''}(s) + \cdots + \frac{1}{4^{q-1}}\, f_{\nu^{(q)}}(s) + \cdots$$

represents a continuous function whose Fourier series with respect to the orthogonal system at hand diverges at the point $s = a$.

The uniform convergence of the series $F(s)$ follows immediately from the fact that all $f_{\nu^{(q)}}$ remain less than 1 in absolute value. To prove the divergence of the Fourier series of $F(s)$ at $s = a$, we show that the *number sequence*

$$[F(a)]_{\nu'}\,,\ [F(a)]_{\nu''}\,,\ [F(a)]_{\nu'''}\,,\ \cdots$$

grows beyond all bounds. To estimate $[F(a)]_{\nu^{(q)}}$, say, we decompose the function $F(s)$ in three summands, as indicated in the formula

$$F(s) = \Big(f_{\nu'}(s) + \cdots + \frac{1}{4^{q-2}}\, f_{\nu^{(q-1)}}(s)\Big) + \frac{1}{4^{q-1}}\, f_{\nu^{(q)}}(s) + \Big(\frac{1}{4^{q}}\, f_{\nu^{(q+1)}}(s) + \cdots\Big)$$

by the inserted parentheses, and consider the $\nu^{(q)\,\mathrm{th}}$ partial sum of the Fourier series of each individual summand at the point $s = a$. The first summand — which in our notation is $F^{(q-1)}(s)$ — contributes in the expression for $[F(a)]_{\nu^{(q)}}$ an amount that by inequality (2) is smaller than $G^{(q-1)}$. The last summand is smaller than $\dfrac{1}{3 \cdot 4^{q-1}}$ in absolute value, and its contributed amount is thus less than $\dfrac{\omega_{\nu^{(q)}}}{3 \cdot 4^{q-1}}$ *). Since finally

$$\big|[f_{\nu^{(q)}}(a)]_{\nu^{(q)}}\big| > \frac{\omega_{\nu^{(q)}}}{2}\,,$$

this implies

$$\big|[F(a)]_{\nu^{(q)}}\big| > \frac{\omega_{\nu^{(q)}}}{2 \cdot 4^{q-1}} - G^{(q-1)} - \frac{\omega_{\nu^{(q)}}}{3 \cdot 4^{q-1}} = \frac{\omega_{\nu^{(q)}}}{6 \cdot 4^{q-1}} - G^{(q-1)}\,.$$

Hence, according to inequality (3),

$$\big|[F(a)]_{\nu^{(q)}}\big| > q - 1\,.$$

Thus our claim is proved.

The condition that the ω_n do not remain below a bound independent of n thus turns out to be *sufficient* for a continuous function to exist whose Fourier series with respect to the considered orthogonal system does not converge. We shall see in the next paragraph that for a very extensive class of orthogonal systems, this condition is also *necessary.*

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*) Indeed, we have that, if $\varphi(s)$ denotes an arbitrary function which on the entire interval $[\alpha, \beta]$ is less than M in absolute value,

$$\big|[\varphi(a)]_{\nu^{(q)}}\big| = \left|\int_{\alpha}^{\beta} K_{\nu^{(q)}}(a,t)\,\varphi(t)\,dt\right| \le M \int_{\alpha}^{\beta} \big|K_{\nu^{(q)}}(a,t)\big|\,dt = M\,\omega_{\nu^{(q)}}\,.$$

§ 2.

Application to Sturm-Liouville Series. *)

The theorem just derived has an immediate application to the theory of Sturm-Liouville series.

If the coefficients $p(x)$ and $q(x)$ of the selfadjoint differential equation

$$L(u) \equiv \frac{d}{dx}\left(p(x)\,\frac{du}{dx}\right) + q\,u + \lambda\,u = 0 \tag{4}$$

are different from zero on the entire interval $[\alpha, \beta]$ (including the boundaries), then the parameter λ can — in infinitely many ways — be determined such that the present equation possesses a solution satisfying the boundary conditions

$$\frac{du}{dx} - h\,u = 0 \quad \text{for} \quad x = \alpha\,, \qquad \frac{du}{dx} + H\,u = 0 \quad \text{for} \quad x = \beta\,. \tag{5}$$

The infinitely many functions

$$u_1(x)\,,\ u_2(x)\,,\ u_3(x)\,,\ \cdots$$

thus obtained form a complete orthogonal function system; we want to call it a *Sturm-Liouville orthogonal system* for short and remark immediately that instead of the boundary conditions (5), an arbitrary pair of homogeneous boundary conditions may be chosen.

To study the orthogonal system $u_n(x)$, we apply to the present differential equation a transformation common to this theory, stemming from Liouville. We put

$$z = \int_\alpha^x \big(p(x')\big)^{-\frac{1}{2}}\,dx' \qquad v(z) = \big(p(x)\big)^{-\frac{1}{4}}\,u(x)\,.$$

Our differential equation then passes into the new differential equation

$$\frac{d^2v}{dz^2} + Q\,u + \lambda\,v = 0\,, \tag{4$'$}$$

where $Q(z)$ stands for a function easily expressible in terms of the functions $p(x)$, $q(x)$.

The boundary conditions become

$$\frac{dv}{dz} - h'\,v = 0 \quad \text{for} \quad z = 0\,, \qquad \frac{dv}{dz} - H'\,v = 0 \quad \text{for} \quad z = \pi\,, \tag{5$'$}$$

where for simplicity, we have assumed

$$\int_\alpha^\beta \big(p(x)\big)^{-\frac{1}{2}}\,dx = \pi$$

— which after all can be obtained always by multiplying the independent variable by a constant; h' and H' are two constants which can be expressed easily in terms

*) With a similar method, Mr. Lebesgue has constructed a continuous function whose trigonometric series is divergent, resp., not uniformly convergent. (Cf. Lebesgue, Séries trigonométriques, p. 87) In a treatise that recently appeared in the Annales de Toulouse (3^{e} série, t. I), Mr. Lebesgue has generalized his results. Quite some common ground with the paper in hand can be found therein.

of the h, H. We denote the Sturm-Liouville functions arising from the differential equation (4′) by

$$v_1(z)\,,\ v_2(z)\,,\ v_3(z)\,,\ \cdots$$

and, to begin with, show that there *exists a continuous function, whose Fourier series with respect to this orthogonal system does not converge.*

To this purpose, we employ an asymptotic representation of the n^{th} term of this function system, due to Liouville and improved by Hobson. *) We assume it to be normalized such that

$$\int_0^\pi \big(v_n(z)\big)^2\,dz = 1\,.$$

Then we have for every point of the interval $[0,\pi]$ that

$$v_n(z) = \sqrt{\frac{2}{\pi}}\,\cos nz\,\big\{1 + \tfrac{\alpha_n(z)}{n^2}\big\} + \sin nz\,\big\{\tfrac{\beta(z)}{n} + \tfrac{\gamma_n(z)}{n^2}\big\}\,,$$

where the functions $\alpha_n(z)$, $\gamma_n(z)$, and $\beta(z)$ remain below a bound A, independent of n and z. To prove the existence of a continuous function whose Fourier series diverges at the point $z = a$, it suffices — according to the theorem derived in § 1 — to show that the quantities

$$\int_0^\pi \big|K_n(a,t)\big|\,dt = \int_0^\pi \big|v_1(a)\,v_1(t) + \cdots + v_n(a)\,v_n(t)\big|\,dt$$

grow beyond all bounds. To this end, we set

$$K_n(a,t) = \frac{2}{\pi}\sum_{p=1,\cdots,n} \cos pa\,\cos pt + \Phi_n(a,t)$$

and prove that $\big|\Phi_n(a,t)\big|$ remains below a bound independent from n, a, and t, but

$$\int_0^\pi \Big|\sum_{p=1,\cdots,n} \cos pa\,\cos pt\Big|\,dt$$

grows beyond all bounds. Namely, if we form $\Phi_n(a,t)$, we obtain firstly the series

$$\sqrt{\frac{2}{\pi}}\Big\{\beta(t)\sum_{p=1,\cdots,n}\frac{\cos pa\,\sin pt}{p} + \beta(a)\sum_{p=1,\cdots,n}\frac{\cos pa\,\sin pt}{p}\Big\} \tag{6}$$

and secondly three finite trigonometric series, whose p^{th} terms have denominator p^2, p^3, p^4, respectively. Since in each term of these latter series, the absolute values of the numerators are less than A^2, these series are certainly less than $A^2 \sum_{p=1,2,\cdots} \frac{1}{p^2}$ in absolute value. To show now also that the series (6) or — which is equivalent — the series

$$\sum_{p=1,\cdots,n}\frac{\cos pa\,\sin pt}{p} \quad\text{and}\quad \sum_{p=1,\cdots,n}\frac{\cos pt\,\sin pa}{p}\,,$$

*) Cf. Hobson, Proceedings of the London Mathematical Society, Ser. 2, Vol. 6 (1908), p. 349.

respectively, remain below a bound independent from n, a, and t, we decompose

$$\sum_{p=1,\cdots,n} \frac{\cos pa\, \sin pt}{p} = \frac{1}{2}\left\{ \sum_{p=1,\cdots,n} \frac{\sin p(t+a)}{p} + \sum_{p=1,\cdots,n} \frac{\sin p(t-a)}{p} \right\},$$

$$\sum_{p=1,\cdots,n} \frac{\cos pt\, \sin pa}{p} = \frac{1}{2}\left\{ \sum_{p=1,\cdots,n} \frac{\sin p(t+a)}{p} + \sum_{p=1,\cdots,n} \frac{\sin p(a-t)}{p} \right\}.$$

Since, however, the series $\sum_{p=1,2,\cdots} \frac{\sin pt}{p}$ is the Fourier series of the function $\frac{\pi-t}{2}$, the sums $\left| \sum_{p=1,\cdots,n} \frac{\sin pt}{p} \right|$ — as taught in the theory of trigonometric series *) — remain below an upper bound independent from t and n, and thus it is shown that $\left|\Phi_n(a,t)\right|$ remains finite.

It remains to prove that the quantities

$$\omega_n = \int_0^\beta \left| \sum_{p=1,\cdots,n} \cos pt\, \cos pa \right| dt$$

become infinitely large as n grows. To this end, we proceed similarly as Mr. Lebesgue does at the place mentioned above.

To shorten the calculations, we assume that the arbitrarily chosen point $z = a$ lies between 0 and $\frac{\pi}{2}$, *i.e.*:

$$0 < \delta < a < \frac{\pi}{2} - \delta \text{ **)}.$$

Now

$$2 \sum_{p=1,\cdots,n} \cos pt\, \cos pa = \frac{\sin(2n+1)\frac{t+a}{2}}{2\sin\frac{t+a}{2}} + \frac{\sin(2n+1)\frac{t-a}{2}}{2\sin\frac{t-a}{2}} - 1\,;$$

since, however, for each value of n and t under consideration, the first summand in this formula in absolute value remains less than the smaller of the two quantities $\left|\frac{1}{2\sin\frac{\delta}{2}}\right|$ and $\left|\frac{1}{2\sin\left(\frac{3\pi}{4}-\frac{\delta}{2}\right)}\right|$, it obviously suffices to show that

$$\omega_n' = \int_0^\pi \left| \frac{\sin\frac{(2n+1)(t-a)}{2}}{\sin\frac{t-a}{2}} \right| dt = \int_{-\frac{a}{2}}^{\frac{\pi-a}{2}} \left| \frac{\sin(2n+1)\vartheta}{\sin\vartheta} \right| d\vartheta$$

becomes infinitely large as n grows. We obviously have

$$\int_{-\frac{a}{2}}^{\frac{\pi-a}{2}} \left| \frac{\sin(2n+1)\vartheta}{\sin\vartheta} \right| d\vartheta > \int_{-\frac{a}{2}}^{\frac{\pi-a}{2}} \left| \frac{\sin(2n+1)\vartheta}{\vartheta} \right| d\vartheta\,.$$

*) Compare, *e.g.*, Kneser, Math. Ann. Vol. 60, p. 402.

**) Should this not be the case, we first would have to modify the integral ω_ν slightly in order to be able to apply our further considerations; however, since we only care to show that there exist continuous functions whose Sturm-Liouville series does not converge, this restriction is irrelevant.

Let us consider the intervals where

$$\left|\sin(2n+1)\vartheta\right| > \sin\frac{\pi}{8} = \mu$$

holds;

$$i_p = \left[\pi\,\frac{\frac{1}{8}+p}{2n+1}, \pi\,\frac{\frac{7}{8}+p}{2n+1}\right]$$

is such an interval; since

$$\int\limits_{(i_p)} \frac{\mu}{\vartheta}\,d\vartheta = \mu\,\log\Big(1+\frac{6}{8p+1}\Big)\,,$$

we certainly have

$$\omega_n' \geq \mu \sum_{p=1,\cdots,\nu} \log\Big(1+\frac{6}{8p+1}\Big)\,,$$

where ν stands for the smallest integer of the property that $\pi\,\dfrac{\frac{7}{8}+\nu}{2n+1} \leq \dfrac{\pi-a}{2}$. Now, however, this number ν — who still depends on n — grows beyond all bounds as n grows; since furthermore the infinite series

$$\sum_{p=1,2,\cdots} \log\Big(1+\frac{6}{8p+1}\Big)$$

diverges *), n can be chosen so large that ω_n becomes larger than an arbitrary number.

This allows us — on account of our general theorem in § 1 — to conclude that there exists a continuous function $F(z)$, whose Fourier series with respect to the $v_n(z)$ diverges at the point $z = a$.

Having proved the existence of this continuous function $F(z)$, it is now easy to show that *the Fourier series of the continuous function*

$$\overline{F}(x) = \big(p(x)\big)^{-\frac{1}{4}}\,F\Bigg(\int\limits_{\alpha}^{x}\big(p(x')\big)^{-\frac{1}{2}}\,dx'\Bigg)$$

with respect to the $u_n(x)$ is divergent. Namely, since by virtue of our substitution

$$v_n(z) = \big(p(x)\big)^{-\frac{1}{4}}\,u_n(x)\,,$$

we have

$$dz = \big(p(x)\big)^{-\frac{1}{2}}\,dx\,,$$

---

*) The easiest way to prove the divergence of this series is to show that the limit of the product $p\,\log\big(1+\frac{6}{8p+1}\big)$ for $p=\infty$ equals $\frac{3}{4}$.

we thus find

$$\int_\alpha^\beta \overline{F}(x)\, u_n(x)\, dx = \int_\alpha^\beta \bigl(p(x)\bigr)^{-\frac{1}{4}} F\biggl(\int_\alpha^x \bigl(p(x')\bigr)^{-\frac{1}{2}}\, dx'\biggr) \bigl(p(x)\bigr)^{-\frac{1}{4}} v_n(z)\, dx$$
$$= \int_0^\pi F(z)\, v_n(z)\, dz\,.$$

Since, however, the series

$$\sum_{n=1,2,\cdots} v_n(z) \int_0^\pi F(z)\, v_n(z)\, dz$$

diverges at the point $z = a$, the same holds for the series

$$\sum_{n=1,2,\cdots} u_n(x) \int_\alpha^\beta \overline{F}(x)\, u_n(x)\, dx = \bigl(p(x)\bigr)^{-\frac{1}{4}} \sum_{n=1,2,\cdots} v_n(z) \int_0^\pi F(z)\, v_n(z)\, dz$$

at the point $x = b$, which by virtue of the transformation

$$z = \int_\alpha^x \bigl(p(x')\bigr)^{-\frac{1}{4}}\, dx'$$

corresponds to the point $z = a$ *). Thus our claim is proved.

§ 3.

Application to Spherical Harmonics.

As n^{th} spherical harmonic or n^{th} Legendre polynomial $P_n(x)$ we denote that solution of the differential equation

$$\frac{d}{dx}\Bigl((1-x^2)\,\frac{dy}{dx}\Bigr) + n(n+1)\,y = 0 \tag{7}$$

which remains finite at the points $x = -1$ and $x = +1$. If n and m differ, we have

$$\int_{-1}^{+1} P_n(x)\, P_m(x)\, dx = 0\,,$$

*) The existence of such a point b for which $a = \int_\alpha^b p(x)^{-\frac{1}{4}}\, dx$, if only $0 < a < \pi$, follows immediately from the fact that the continuous function $z = \int_\alpha^x p(x')^{-\frac{1}{4}}\, dx'$ at the points $x = \alpha$ and $x = \beta$ assumes the values 0 and π, respectively; therefore there has to exist a point b between α and β where it assumes the value a.

and it is common to normalize the $P_n(x)$ in such a way that

$$\int_{-1}^{+1} \big(P_n(x)\big)^2\, dx = \frac{2}{2\,n+1}\,.$$

The system of Legendre polynomials is not a Sturm-Liouville orthogonal system, since in the differential equation (7), the coefficient of $\frac{d^2y}{dx^2}$ vanishes at the points $x = 1$ and $x = -1$. We want to show that there exist continuous functions whose spherical harmonic series diverges.

If we put, as before,

$$K_n(x,t) = \sum_{p=0,1,2,\cdots,n} \frac{2\,p+1}{2}\, P_p(x)\, P_p(t)\,,$$

then the n^{th} partial sum of the spherical harmonic series of a function $f(x)$ is given by

$$[f(x)]_n = \int_{-1}^{+1} K_n(x,t)\, f(t)\, dt\,,$$

and thus we have to show that the infinitely many quantities $\int_{-1}^{+1} \big|K_n(x,t)\big|\, dt$ do not remain below a bound independent of n. We show this for the point $x = 0$.

A well-known formula states *) that

$$K_n(x,t) = \frac{n+1}{2}\, \frac{P_{n+1}(x)\, P_n(t) - P_n(x)\, P_{n+1}(t)}{x-t}\,;$$

and since $P_n(0)$ vanishes for each odd index, it suffices to show that the quantities

$$\omega_{2n} = \frac{2\,n+1}{2}\, \big|P_{2n}(0)\big| \int_{-1}^{+1} \left|\frac{P_{2n+1}(t)}{t}\right| dt$$

with increasing n grow arbitrarily large. To this end, we apply the frequently used approximation formula

$$P_n(\cos\theta) = \sqrt{\frac{2}{n\,\pi\,\sin\theta}}\left[\cos\!\left(\Big(n+\frac{1}{2}\Big)\theta - \frac{\pi}{4}\right) + \frac{\alpha_n(\theta)}{n}\right],$$

which within the interval $[-1+\varepsilon, 1-\varepsilon]$ represents the spherical harmonics for each value of the index n, where ε stands for a positive number different from zero; the functions $\alpha_n(\theta)$ remain below a bound independent of n and θ, as $\cos\theta$ varies within the interval $[-1+\varepsilon, 1-\varepsilon]$. This formula shows immediately that

$$\underset{n=\infty}{L} \sqrt{n\,\pi}\, \big|P_{2n}(0)\big| = 1\,,$$

*) Cf. Christoffel, Journal für Mathematik Vol. 55, p. 73.

and from this it follows that the ω_{2n} will certainly grow beyond all bounds, if the quantities

$$\sqrt{2\,n+1}\int_{-1}^{+1}\left|\frac{P_{2n+1}(t)}{t}\right|dt > \sqrt{2\,n+1}\int_{-\frac{1}{\sqrt{2}}}^{0}\left|\frac{P_{2n+1}(t)}{t}\right|dt = \omega'_{2n}$$

become infinitely large with growing n. To show this, we put

$$t = \cos\Big(\vartheta + \frac{\pi}{2}\Big) = -\sin\vartheta\,.$$

We obtain

$$\omega'_{2n} = \sqrt{2\,n+1}\int_0^{\frac{\pi}{4}}\left|\frac{P_{2n+1}\big(\cos(\vartheta+\frac{\pi}{2})\big)}{\sin\vartheta}\cos\vartheta\right|d\vartheta\,;$$

since, however, in the entire interval $\Big[0,\frac{\pi}{4}\Big]$, we have

$$\cos\vartheta \geq \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\vartheta \leq \vartheta\,,$$

we thus find

$$\omega'_{2n} > \sqrt[4]{2}\int_0^{\frac{\pi}{4}}\left|\frac{\sqrt{\frac{2n+1}{2}\cos\vartheta}\,P_{2n+1}\big(\cos(\vartheta+\frac{\pi}{2})\big)}{\vartheta}\right|d\vartheta\,.$$

However, according to our approximation formula, we have

$$\begin{aligned}
&\sqrt{\frac{2n+1}{2}\cos\vartheta}\;P_{2n+1}\Big(\cos\big(\vartheta+\frac{\pi}{2}\big)\Big) = \\
&\qquad = \frac{1}{\sqrt{\pi}}\left(\cos\Big(\big(2\,n+\frac{3}{2}\big)\big(\vartheta+\frac{\pi}{2}\big)-\frac{\pi}{4}\Big)+\frac{\alpha_{2n+1}(\vartheta+\frac{\pi}{2})}{2\,n+1}\right)\\
&\qquad = \frac{1}{\sqrt{\pi}}\left((-1)^{n+1}\sin\big(2\,n+\frac{3}{2}\big)\vartheta+\frac{\alpha_{2n+1}(\vartheta+\frac{\pi}{2})}{2\,n+1}\right).
\end{aligned}$$

Let us write $\sin\dfrac{\pi}{8} = \mu$ for short, and choose from now on n so large that we have $\left|\dfrac{\alpha_{2n+1}(\vartheta+\frac{\pi}{2})}{2\,n+1}\right| < \dfrac{\mu}{2}$ for each value of ϑ in the interval $\Big[0,\dfrac{\pi}{4}\Big]$. If ϑ is enclosed between the bounds $\pi\,\dfrac{\frac{1}{8}+p}{2\,n+\frac{3}{2}}$ and $\pi\,\dfrac{\frac{7}{8}+p}{2\,n+\frac{3}{2}}$, where p denotes an integer, we certainly have

$$\left|\sin\Big(2\,n+\frac{3}{2}\Big)\vartheta\right| > \mu\,,$$

and since $\left|\dfrac{\alpha_{2n+1}(\vartheta+\frac{\pi}{2})}{2\,n+1}\right| < \dfrac{\mu}{2}$, we thus find

$$\left|\sqrt{\frac{2\,n+1}{2}\cos\vartheta}\;P_{2n+1}\Big(\cos\big(\vartheta+\frac{\pi}{2}\big)\Big)\right| > \frac{\mu}{2\sqrt{\pi}}\,.$$

Consequently, we have for the integral taken between these bounds

$$\int\limits_{(i_p)} \left| \frac{\sqrt{\frac{2n+1}{2}\ \cos\vartheta}\, P_{2n+1}\big(\cos(\vartheta+\frac{\pi}{2})\big)}{\vartheta} \right| d\vartheta > \frac{\mu}{2\sqrt{\pi}}\ \log\Big(1+\frac{6}{8\,p+1}\Big)\,.$$

Now, however, the intervals $\left[\pi\,\frac{\frac{1}{8}+p}{2\,n+\frac{3}{2}},\,\pi\,\frac{\frac{7}{8}+p}{2\,n+\frac{3}{2}}\right]$ lie within the interval $\left[0,\,\frac{\pi}{4}\right]$ as soon as $0 \le p \le \frac{n-1}{2}$, and thus we obtain

$$\omega'_{2n} > \frac{\sqrt[4]{2}\,\mu}{2\sqrt{\pi}} \sum_{p=1,2,\cdots,\frac{n-1}{2}} \log\Big(1+\frac{6}{8\,p+1}\Big)\,.$$

From the divergence of the infinite series $\sum\limits_{p=1,\cdots} \log\Big(1+\frac{6}{8\,p+1}\Big)$, however, it follows immediately that the ω'_{2n} grow beyond all bounds, and thus our claim is proved.

Our general criterion is applicable without any difficulties also in such cases where the orthogonal series are supposed to be *“summed up”* with some *summation method*. By this we understand the following: the infinitely many functions

$$a_1(n),\ a_2(n),\ a_3(n),\ \cdots$$

defined on the point set M have the property that for a certain value $n = n_0$, say, which is an accumulation point of the point set M, we have

$$\underset{n=n_0}{L}\ a_p(n) = 1 \qquad\qquad (p = 1, 2, 3, \cdots)\,.$$

Let us now consider an arbitrary infinite series

$$u_1 + u_2 + u_3 + \cdots\,,$$

which has the property, though, that the series

$$a_1(n)\,u_1 + a_2(n)\,u_2 + a_3(n)\,u_3 + \cdots$$

converges for each value of n belonging to the point set M. If now the limit

$$S = \underset{n=n_0}{L}\ \big(a_1(n)\,u_1 + a_2(n)\,u_2 + a_3(n)\,u_3 + \cdots\big)$$

exists, we say that *the present series is summable with the aid of the summation method given by the functions $a_p(n)$, and assign S to it as its “sum”.*

If the infinite series

$$K(n;\,a,t) = a_1(n)\,\varphi_1(a)\,\varphi_1(t) + a_2(n)\,\varphi_2(a)\,\varphi_2(t) + \cdots$$

converge for each value of n under consideration, then the investigations of § 1 yield the following theorem: *if we have*

$$\text{lim. sup.} \int\limits_{\alpha}^{\beta} \big|K(n;\,a,t)\big|\,dt = \infty\,,$$

then it is possible to specify a continuous function, whose Fourier series (*with respect to the* $\varphi_p(s)$) *at the point* $s = a$ *is not summable with the aid of the summation method given by the functions* $a_p(n)$.

Chapter II.

Theory of Summation.

If

$$\varphi_1(s),\ \varphi_2(s),\ \cdots,\ \varphi_n(s),\ \cdots$$

is an arbitrary orthogonal function system defined on the interval $[\alpha, \beta]$, then we say that a function $f(s)$ defined on the interval $[\alpha, \beta]$ belongs to the "*range*" of this function system, if for each arbitrarily small number δ, there can be determined n constants $c_1, \cdots, c_n$ such that in the entire interval, we have

$$\left| f(s) - c_1\,\varphi_1(s) - c_2\,\varphi_2(s) - \cdots - c_n\,\varphi_n(s) \right| < \delta\,.$$

The set of these functions $f(s)$ form the *range* of the present orthogonal system. This notion plays an exceedingly important rôle in summation theory, since the Fourier series can only be summed up for functions in the range in such a way that the sequence of functions arising from the summation be uniformly convergent. By the by, this notion of range is a very comprehensive one in the known examples; for instance, for the trigonometric functions, for the Legendre polynomials, or even for an arbitrary orthogonal system arising from a differential equation, it consists of all continuous functions, which if need be satisfy certain boundary conditions. In general, we can say: if all *analytical* functions can be expanded with respect to the functions of an orthogonal system, then all *continuous* functions of the interval — by virtue of the well-known Weierstraß theorem — belong to the range of this function system.

§ 1.

A Lemma.

We have proved in Chapter I that if the present orthogonal function system satisfies a certain condition, it is always possible to specify a continuous function whose Fourier series with respect to this system diverges at a given point. We shall show now that for those orthogonal systems whose range comprises all continuous functions, this condition is also *necessary* for such a function to exist. To this end, we prove the following simple lemma *):

*) As a special case of this lemma, there arises a theorem pronounced by Mr. H. Lebesgue (Rendiconti del Circolo matematico di Palermo vol. 26 (1908), p. 325).

To each function $f(s)$ of a certain class of functions, let there be assigned a sequence of real functions $f_1(s),\ f_2(s),\ \cdots$; in symbols

$$f(s) \sim f_1(s),\ f_2(s),\ \cdots .$$

This assignment shall have the following properties:

A) *If*

$$f(s) \sim f_1(s),\ f_2(s),\ \cdots$$

and

$$g(s) \sim g_1(s),\ g_2(s),\ \cdots ,$$

then

$$f(s)+g(s) \sim f_1(s)+g_1(s),\ f_2(s)+g_2(s),\ \cdots .$$

B) *For each s, let $\left|f_p(s)\right|$ always be less than the upper bound of $\left|f(s)\right|$ multiplied by a quantity M, which for all functions of the class is the same: $\left|f_p(s)\right| < M \cdot \operatorname{Max}\left|f(s)\right|$.*

If now $f'(s),\ f''(s),\ \cdots$ are a sequence of functions which converge to the function $f(s)$ uniformly in s, and if the function sequences assigned to $f^{(n)}(s)$ by virtue of our assignment, converge to $F^{(n)}(s)$, respectively, uniformly in s, i.e., *there prevail the limit equations*

$$\underset{p=\infty}{L} f_p^{(n)}(s) = F^{(n)}(s) \tag{8}$$

uniformly in s, then the function sequence assigned to $f(s)$ converges uniformly to a function $F(s)$, and we have

$$\underset{n=\infty}{L} F^{(n)}(s) = F(s)\,.$$

Namely, the convergence of the sequence $f'(s),\ f''(s),\ \cdots$ to the function $f(s)$ implies that for sufficiently large q and q', we have

$$\left|f^{(q)}(s) - f^{(q')}(s)\right| < \varepsilon\,,$$

however small ε be chosen; since furthermore, by our first assumption, the sequence assigned to the function $f^{(q)}(s)-f^{(q')}(s)$ consists of the differences $f_p^{(q)}(s)-f_p^{(q')}(s)$, it follows from the second assumption that

$$\left|f_p^{(q)}(s) - f_p^{(q')}(s)\right| < \varepsilon\, M \tag{9}$$

for sufficiently large q and q' and arbitrary p. Since, however, the limit equations (8) prevail, we can, after fixing the indices q, q', still choose the index p so large that

$$\left|F^{(q')}(s) - f_p^{(q')}(s)\right| < \varepsilon \quad \text{and} \quad \left|F^{(q)}(s) - f_p^{(q)}(s)\right| < \varepsilon\,.$$

From the last three inequalities, however, it follows by addition that

$$\left|F^{(q)}(s) - F^{(q')}(s)\right| < (M+2)\,\varepsilon\,,$$

and this inequality states that *the functions $F^{(n)}(s)$ converge uniformly to a function $F(s)$.*

In order to show that this function $F(s)$ is the uniform limit of the function sequence $f_1(s)$, $f_2(s)$, $\cdots$ assigned to $f(s)$, we note that for sufficiently large n and arbitrary p, it turns out that

$$\left|f_p^{(n)}(s) - f_p(s)\right| < \varepsilon\, M\,.$$

This follows from our assumptions A) and B) just like the inequality (9) derived above. Moreover, we choose n so large that

$$\left|F(s) - F^{(n)}(s)\right| < \varepsilon\,,$$

and then determine for this fixed index n a quantity P such that

$$\left|F^{(n)}(s) - f_p^{(n)}(s)\right| < \varepsilon$$

whenever $p > P$. By adding the last three inequalities we realize that

$$\left|F(s) - f_p(s)\right| < (M+2)\,\varepsilon$$

for any sufficiently large p; and so our theorem is proved *).

We want to draw immediately an important conclusion from this theorem.

We assign to the function $f(s)$ the functions $f_p(s)$ which in the entire interval $[\alpha, \beta]$ have the value that the p^{th} partial sum of the Fourier series of $f(s)$, formed with respect to the orthogonal system $\varphi_1(s)$, $\varphi_2(s)$, $\cdots$, assumes at an arbitrary point $s = a$, say:

$$f_p(s) = [f(a)]_p = \int\limits_\alpha^\beta K_p(a,t)\, f(t)\, dt\,.$$

This assignment obviously satisfies condition A). If we furthermore do have

$$\int\limits_\alpha^\beta \left|K_p(a,t)\right| dt < M\,,$$

though, where M denotes a number independent of p, then also our second assumption B) is satisfied. If we understand by $\varphi(s)$ any finite aggregate

$$\varphi(s) = a_1\, \varphi_1(s) + \cdots + a_n\, \varphi_n(s)$$

of our orthogonal functions, then obviously the function sequence assigned to $\varphi(s)$ converges uniformly to the value of this function at the point $s = a$. Is now $f(s)$ an arbitrary function in the range of our orthogonal system, we can pick out a sequence $\varphi'(s)$, $\varphi''(s)$, $\cdots$ of the functions $\varphi(s)$ considered just now, converging uniformly to $f(s)$. According to the lemma just now proved, the sequence

$$[f(a)]_1,\ [f(a)]_2,\ \cdots$$

*) It is worth noting that in the proof of this theorem, the circumstance that the occuring functions depend on only one variable was not made use of at all. The theorem thus remains correct if all functions depend on several independent variables.

assigned to $f(s)$ then also has to converge to $f(a)$. With that, though, it is shown that *if we have*

$$\int_{\alpha}^{\beta} \left|K_p(a,t)\right| dt < M \qquad (p = 1, 2, 3, \cdots),$$

then the Fourier series of each function belonging to the range of this orthogonal system converges at the point $s = a$. This theorem tells us that in the quite general case that the range of our orthogonal system contains all continuous functions, the sufficient condition given on p. 5 for the existence of a continuous function which can not be expanded with respect to this orthogonal system is also *necessary.*

§ 2.

Application to the Theory of Trigonometric and Sturm-Liouville Series.

Out of consideration for the subsequent explanations, we want to derive in advance two theorems from the classical theory of trigonometrics series, based on our lemma p. 18:

1) The so-called Poisson integral

$$f_r(s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2\,r\cos(s-t) + r^2}\, f(t)\, dt$$

assigns to each function $f(s)$ a function set $f_r(s)$. This assignment obviously satisfies condition A) of our lemma. Since, however,

$$\frac{1 - r^2}{1 - 2\,r\cos(s-t) + r^2}$$

is always positive whenever $r < 1$, and since consequently we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1 - r^2}{1 - 2\,r\cos(s-t) + r^2} \right| dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + 2 \sum_{n=1,2,\cdots} r^n \cos n(s-t) \right\} dt = 1$$

for any value of r and s taken into consideration, it follows that $f_r(s)$ taken in absolute value is less than the maximum of $|f(s)|$, no matter how r and s are chosen. In other words, the assignment given by the Poisson integral also satisfies assumption B). As is well known, we have for every value $r < 1$

$$f_r(s) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$

$$+\ \frac{1}{\pi}\sum_{n=1,2,\cdots} r^n\left\{\cos ns\int_0^{2\pi} f(t)\,\cos nt\,dt + \sin ns\int_0^{2\pi} f(t)\,\sin nt\,dt\right\},$$

where the series on the right for $r < 1$ converges absolutely and uniformly. From this we see that the function sets assigned to the functions $\cos ns$ and $\sin ns$, respectively, converge uniformly to these functions, as the parameter r converges to 1. An immediate consequence thereof is the fact that if $\Phi(s)$ stands for an arbitrary trigonometric polynomial:

$$\Phi(s) = a_0 + a_1\,\cos s + a_1'\,\sin s + \cdots + a_n\,\cos ns + a_n'\,\sin ns\,,$$

then the function set assigned to it converges uniformly to $\Phi(s)$. Does $F(s)$ now denote any continuous function with period 2π, we can pick out from the trigonometric polynomials a sequence $\Phi'(s)$, $\Phi''(s)$, $\cdots$ which converges uniformly to $F(s)$. Our lemma states, however *), that then also *the functions*

$$F_r(s) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2\,r\cos(s-t)+r^2}\,F(t)\,dt$$

assigned to $F(s)$ converge uniformly to $F(s)$ for $r = 1$, if $F(s)$ denotes a continuous periodic function.

2) *The Fejér summation method of trigonometric series.* Is $F(s)$ a periodic function, then, as is well known, the series

$$[F(s)]_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\frac{2n+1}{2}(s-t)}{\sin\frac{s-t}{2}}\,F(t)\,dt$$

need not converge. If we put

$$[F^*(s)]_n = \frac{[F(s)]_0 + [F(s)]_1 + \cdots + [F(s)]_{n-1}}{n}\,,$$

though, then the sequence of these $[F^*(s)]_n$, as Mr. Fejér has shown, converges uniformly to $F(s)$. Indeed, we find by a simple calculation

$$[F^*(s)]_n = \frac{1}{2n\pi}\int_0^{2\pi}\frac{\sin^2 n\frac{s-t}{2}}{\sin^2\frac{s-t}{2}}\,F(t)\,dt\,.$$

*) The fact that the assigned function set is not countable, but of the cardinality of the continuum, is obviously irrelevant.

By this formula, each function $F(s)$ is assigned the function sequence $[F^*(s)]_n$; this assignment satisfies assumption A), and since we have

$$\frac{1}{2n\pi}\int_0^{2\pi}\frac{\sin^2 n\frac{s-t}{2}}{\sin^2\frac{s-t}{2}}\,dt = 1\,,$$

assumption B) is satisfied also. It is also easy to see that the function sequences assigned to the trigonometric polynomials converge uniformly to these functions. From this, there follows, by our lemma and the Weierstraß theorem mentioned above, the theorem by Mr. Fejér.

3) Finally, we make one more application of our lemma to the *convergence theory of Sturm-Liouville series.*

Keeping to the terminology of the chapter before (p. 10), we consider the Sturm-Liouville orthogonal system $v_1(z)$, $v_2(z)$, $\cdots$ treated there and again put

$$K_n(z,t) = v_1(z)\,v_1(t) + \cdots + v_n(z)\,v_n(t)\,.$$

We have proved (p. 10) that the difference

$$\Phi_n(z,t) = K_n(z,t) - \frac{2}{\pi}\sum_{p=1,\cdots,n}\cos pz\,\cos pt$$

in absolute value remains below an upper bound Φ independent of n, z, and t; from this we conclude that the assignment given by the integral

$$f_n(z) = \int_0^{\pi}\left\{K_n(z,t) - \frac{2}{\pi}\sum_{p=1,\cdots,n}\cos pz\,\cos pt\right\} f(t)\,dt = \int_0^{\pi}\Phi_n(z,t)\,f(t)\,dt \tag{10}$$

satisfies the assumptions of our lemma. If $\varphi(z)$ now denotes some *analytic* function, thus belonging to the range of *both* orthogonal systems

$$1,\ \cos z,\ \cos 2z,\ \cdots\,,\quad\text{and}\quad v_1(z),\ v_2(z),\ v_3(z),\ \cdots\,,$$

then, as we know, both integrals

$$\frac{2}{\pi}\int_0^{\pi}\sum_{p=1,\cdots,n}\cos pz\,\cos pt\,\varphi(t)\,dt \quad\text{and}\quad \int_0^{\pi}K_n(z,t)\,\varphi(t)\,dt$$

converge with growing n uniformly to $\varphi(z)$, since the Fourier series of these functions with respect to both orthogonal systems converge uniformly. Consequently, the integral

$$\int_0^{\pi}\Phi_n(z,t)\,\varphi(t)\,dt$$

has limit zero. Our lemma now states that the integral (10) also converges to zero, if $f(z)$ is an arbitrary function that can be approximated uniformly by the functions $\varphi(z)$. In other words, our integral converges to zero if $f(z)$ is an arbitrary continuous function *). This immediately implies the following theorem:

*) Namely, if we assume the boundary conditions (5) or (5′), respectively, the range of the function system $v_1(z)$, $v_2(z)$, $\cdots$ comprises *all* continuous functions.

Let $f(z)$ be an arbitrary continuous function; *the Fourier series of this function with respect to the orthogonal system $v_1(z)$, $v_2(z)$, $\cdots$ converges or diverges, respectively, according as the cosine series of this function is convergent or divergent.*

Is now $F(z)$ an *arbitrary* function, integrable in the Lebesgue sense, then we construct a sequence of *continuous* functions: $f'(z)$, $f''(z)$, $\cdots$ such that we have

$$\underset{p=\infty}{L} \int_0^\pi \left|F(z) - f^{(p)}(z)\right| dz = 0\,.$$

Then we obviously have for each p

$$\left|\int_0^\pi \Phi_n(z,t)\, F(t)\, dt\right| \leq \Phi \int_0^\pi \left|F(t) - f^{(p)}(t)\right| dt + \left|\int_0^\pi \Phi_n(z,t)\, f^{(p)}(t)\, dt\right|.$$

From this inequality, we deduce by means of the theorem just proved that we have

$$\underset{n=\infty}{L} \int_0^\pi \Phi_n(z,t)\, F(t)\, dt = 0\,,$$

*i.e., the Sturm-Liouville expansion of an integrable function is convergent or divergent at some point, respectively, according as the cosine series of this function at this point is convergent or divergent**).

Also, it is immediate how these theorems are to be modified in the case that instead of the boundary conditions (5) (p. 9), some other pair of homogeneous boundary conditions is assumed.

§ 3.

Summation of Sturm-Liouville Series.

We now want to show that the summation method applied to the trigonometric series by Mr. Fejér can be applied to the Sturm-Liouville series with equal success.

We keep to the terminology from § 2 of the first chapter, and consider on the interval $[\alpha, \beta]$ the eigenfunctions $u_1(x)$, $u_2(x)$, $\cdots$ of the differential equation

$$L(u) \equiv \frac{d}{dx}\left(p\,\frac{du}{dx}\right) + q\,u + \lambda\,u = 0\,, \tag{4}$$

$$(p(x) > 0 \text{ in the interval } [\alpha, \beta])$$

with a pair of homogeneous boundary conditions

$$\frac{du}{dx} - h\,u = 0 \quad \text{for} \quad x = \alpha \qquad \text{and} \qquad \frac{du}{dx} + H\,u = 0 \quad \text{for} \quad x = \beta\,. \tag{5}$$

The eigenfunctions of the differential equation arising from (4) by virtue of the Liouville transformation shall again be denoted $v_1(z)$, $v_2(z)$, $\cdots$. Because of the relations derived on p. 12 between the two orthogonal systems $u_n(x)$ and $v_n(z)$, it

*) This remark allows for a new proof of the theorem that there exist continuous functions whose Sturm-Liouville series diverge.

obviously suffices to restrict oneself to this latter system, since the results obtained for this are transferable to the function system $u_n(x)$ without any difficulty.

Let $f(z)$ be an arbitrary function with Fourier series

$$\sum_{n=1,2,\cdots} v_n(z) \int_0^\pi f(t)\, v_n(t)\, dt\,;$$

we consider the *arithmetic means formed from the partial sums of this series*

$$[f^*(z)]_1 = [f(z)]_1,\ [f^*(z)]_2 = \frac{[f(z)]_1 + [f(z)]_2}{2},$$
$$[f^*(z)]_3 = \frac{[f(z)]_1 + [f(z)]_2 + [f(z)]_3}{3},\ \cdots .$$

If we now put in the same manner as before

$$K_n^*(z,t) = \frac{K_1(z,t) + K_2(z,t) + \cdots + K_n(z,t)}{n}\,, \tag{11}$$

then we have

$$[f^*(z)]_n = \int_0^\pi K_n^*(z,t)\, f(t)\, dt\,.$$

If we now assign to each function $f(z)$ the functions

$$[f^*(z)]_1,\ [f^*(z)]_2,\ [f^*(z)]_3,\ \cdots$$

so defined, this assignment obviously satisfies the assumption A) of our lemma p. 18. Do we furthermore have $f(z) = v_n(z)$, then the function sequence assigned to $v_n(z)$ is given by:

$$0,\ 0,\ \cdots,\ 0,\ \frac{v_n(z)}{n},\ \frac{2\, v_n(z)}{n+1},\ \cdots,\ \frac{p\, v_n(z)}{n+p},\ \cdots$$

and we see that this function sequence converges uniformly to $v_n(z)$ for $p = \infty$. From this, it also follows immediately that if $v(z)$ denotes a finite aggregate of the form

$$v(z) = a_1\, v_1(z) + \cdots + a_n\, v_n(z)$$

(with constant coefficients a), then the function sequence assigned to this function converges uniformly to $v(z)$. If this assignment now satisfies assumption B) of our lemma also, we may conclude therefrom that *the Fourier series of each function lying in the range of the orthogonal function system $v_n(z)$ is summable by the method of the arithmetic mean.*

To show this, it obviously suffices though to prove that the integral

$$\int_0^\pi \left|K_n^*(z,t)\right| dt$$

remains below a bound M independent of n and z. Now we do have though (cf. p. 10)

$$K_n(z,t) = \frac{2}{\pi} \sum_{p=1,\cdots,n} \cos pz\, \cos pt + \Phi_n(z,t)\,,$$

and we have shown at that place that $\Phi_n(z,t)$ remains below a bound independent of n, z, and t. Since we have, though,

$$K_n^*(z,t) = \frac{\sum\limits_{p=1} \cos pz\, \cos pt + \sum\limits_{p=1,2} \cos pz\, \cos pt + \cdots + \sum\limits_{p=1,\cdots,n} \cos pz\, \cos pt}{n} + \frac{\Phi_1(z,t) + \cdots + \Phi_n(z,t)}{n}\,,$$

this implies that $\int\limits_0^\pi \left|K_n^*(z,t)\right| dt$ definitely lies below a finite bound, since we have

$$\begin{aligned}
&\int\limits_0^\pi \left| \frac{\sum\limits_{p=1} \cos pz\, \cos pt + \cdots + \sum\limits_{p=1,\cdots,n} \cos pz\, \cos pt}{n} \right| dt \\
&\qquad = \frac{1}{2\,n} \int\limits_0^\pi \left| -(n+1) + \frac{1}{2} \left\{ \frac{\sin^2 (n+1) \frac{z+t}{2}}{\sin^2 \frac{z+t}{2}} + \frac{\sin^2 (n+1) \frac{z-t}{2}}{\sin^2 \frac{z-t}{2}} \right\} \right| dt \\
&\qquad \le \frac{n+1}{2\,n}\,\pi + \frac{1}{4\,n} \int\limits_0^\pi \left| \frac{\sin^2 (n+1) \frac{z+t}{2}}{\sin^2 \frac{z+t}{2}} + \frac{\sin^2 (n+1) \frac{z-t}{2}}{\sin^2 \frac{z-t}{2}} \right| dt \\
&\qquad = \frac{n+1}{2\,n}\,\pi + \frac{n+1}{4\,n}\,\pi = \frac{3(n+1)}{4\,n}\,\pi\,,
\end{aligned}$$

and $\frac{\Phi_1(z,t)+\cdots+\Phi_n(z,t)}{n}$ remains less than a number independent of n, z, t.

Thus it is shown that the second assumption of our lemma is fulfilled also, and we obtain the result *):

The Sturm-Liouville expansion of a function in the range of the Sturm-Liouville orthogonal system under consideration is always summable by the method of the arithmetic mean.

If we assume the boundary conditions (5), then the range of the orthogonal system under consideration consists of all continuous functions (since all twice differentiable functions satisfying the boundary condition (5) are expandable); this implies the following theorem:

If a continuous function is expanded in the Fourier manner into a series that progresses according to the eigenfunctions of the differential equation (4), *satisfying the boundary conditions* (5), *then the sequence* $[f^*(x)]_n$ *of arithmetic means converges uniformly to the function* $f(x)$ **).

If instead of the boundary conditions (5), a different pair of conditions is assumed,

$$u(\alpha) = 0 \qquad \text{and} \qquad u(\beta) = 0\,, \tag{12}$$

say, then in the very same way the theorem can be proved that the sequence of arithmetic means $[f^*(x)]_n$ converges uniformly to $f(x)$, if $f(x)$ lies in the range of the $u_n(x)$. In this case, though, we have to bear in mind that the range is now made up from the continuous functions vanishing at the points $x = \alpha$ and $x = \beta$, *i.e.*: *if a*

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*) Of course, this theorem can also be deduced from the theorem p. 23, but it seems expedient to take this very generalizable course.

**) An immediate consequence of this theorem is the fact that if the Sturm-Liouville series of a continuous function $f(s)$ converges at the point $s = a$, then its sum equals $f(a)$.

continuous function satisfying the boundary condition (12) *is expanded with respect to the eigenfunctions of the differential equation* (4) *which satisfy the boundary condition* (12), *then this series is according to the usual terminology "Cesàro summable"*, i.e., *the sequence of arithmetic means* $\frac{[f(z)]_1 + \cdots + [f(z)]_n}{n}$ *formed from the partial sums* $[f(z)]_n$ *converges uniformly to the function* $f(z)$. The theorem is to be modified accordingly, if some other pair of homogeneous boundary conditions is assumed.

§ 4.

Generalizations.

The investigations of this chapter are applicable immediately to the expansion of functions with several variables, since our lemma — which constitutes the sole basis for the proofs of this paragraph — remains true also in these more general cases. (Cf. p. 19.)

If we terminate the formal Fourier series of a function $f(s, \sigma)$ with respect to the orthogonal system $\varphi_1(s)$, $\varphi_2(s)$, $\cdots$

$$\sum_{(p)} \sum_{(q)} \varphi_p(s)\, \varphi_q(\sigma) \int_\alpha^\beta \int_\alpha^\beta f(t, \tau)\, \varphi_p(t)\, \varphi_q(\tau)\, dt\, d\tau \tag{13}$$

at the term with indices n, m, we denote the finite sum thus obtained by $[f(s,\sigma)]_{n,m}$:

$$[f(s,\sigma)]_{n,m} = \sum_{p=1,\cdots,n} \; \sum_{q=1,\cdots,m} \varphi_p(s)\, \varphi_q(\sigma) \int_\alpha^\beta \int_\alpha^\beta f(t, \tau)\, \varphi_p(t)\, \varphi_q(\tau)\, dt\, d\tau\,.$$

In our terminology (p. 5), we have

$$[f(s,\sigma)]_{n,m} = \int_\alpha^\beta \int_\alpha^\beta K_n(s,t)\, K_m(\sigma,\tau)\, f(t,\tau)\, dt\, d\tau\,. \tag{13$'$}$$

If this assignment satisfies the conditions of our lemma, we can conclude that the expansion of each function of two variables lying in the range of our orthogonal system converges uniformly to this function.

Let us return now to the Sturm-Liouville function systems; it is easily shown that for this system, equation (13′) does not satisfy assumption B) of our lemma. The assignment

$$[f^*(s,\sigma)]_{n,n} = \int_0^\pi \int_0^\pi K_n^*(s,t)\, K_n^*(\sigma,\tau)\, f(t,\tau)\, dt\, d\tau\,, \tag{14}$$

where $K_n^*(s,t)$ denotes the function defined by equation (11) p. 24, does satisfy both assumptions of our lemma, though, since we have shown that

$$\int_0^\pi \left|K_n^*(s,t)\right| dt$$

lies below an upper bound independent of n, s, and t. This immediately implies that the function sequence defined by (14) *converges uniformly to $f(s,\sigma)$, if this function lies in the range of the Sturm-Liouville system under consideration.*

To this fact there corresponds the following summation method: *from the partial sums $[f]_{n,m}$ of the doubly infinite series* (13), *form the simple sequence:*

$$[f^*]_{n,n} = \frac{1}{n^2} \sum_{p=1,\cdots,n} \sum_{q=1,\cdots,n} [f]_{p,q} \qquad (n = 1, 2, \cdots). \tag{15}$$

If the function $f(s,\sigma)$ lies in the range of the Sturm-Liouville function system under consideration, then the sequence (15) *converges uniformly to $f(s,\sigma)$* *).

Finally, we make one more application of our lemma to the general theory of the summation of orthogonal series. Let there be given a summation method by the infinitely many functions

$$a_1(n),\ a_2(n),\ a_3(n),\ \cdots\ ;$$

if the sum:

$$K(n;a,t) = \sum_{p=1,2,\cdots} a_p(n)\,\varphi_p(a)\,\varphi_p(t)$$

converges, then the necessary and sufficient condition for the Fourier series of each function belonging to the range of the present orthogonal system to be "summable" at the point a with the aid of this summation method, is that the integral

$$\int_\alpha^\beta \left|K(n;a,t)\right| dt$$

remains below an upper bound independent of n.

Chapter III.

On a Class of Orthogonal Function Systems.

The purpose of this paragraph is to treat a class of orthogonal function systems who, besides a series of other noteworthy properties, are particularly distinguished by the fact that the *Fourier series* with respect to these systems *of each continuous function converge and represent the function.* In §§ 1–3 we consider the most simple representative of this class; § 4 will then present the generalization of the theorems we gained to further systems.

*) Instead of the simple sequence $[f^*]_{n,n}$, we could also define a double sequence $[f^*]_{n,m}$ of the same property, but this "simple" summation is to be preferred to the other one.

§ 1.

The Orthogonal Function System χ.

The complete orthogonal function system χ, the most simple representative of that class of orthogonal systems, we define as follows:

Let $\chi_0(s) = 1$ on the entire interval $[0,1]$ including the boundaries; thereupon let:

$$\begin{aligned}\chi_1(s) &= 1 \quad \text{for} \quad 0 \leq s < \frac{1}{2}\,,\\ &= -1 \quad \text{for} \quad \frac{1}{2} < s \leq 1\,.\end{aligned}$$

We furthermore put:

$$\begin{array}{rlrll}\chi_2^{(1)}(s) = & \sqrt{2} \quad \text{and} & \chi_2^{(2)}(s) = & 0 & \text{for} \quad 0 \leq s < \frac{1}{4}\,,\\ = & -\sqrt{2} & = & 0 & ,, \quad \frac{1}{4} < s < \frac{1}{2}\,,\\ = & 0 & = & \sqrt{2} & ,, \quad \frac{1}{2} < s < \frac{3}{4}\,,\\ = & 0 & = & -\sqrt{2} & ,, \quad \frac{3}{4} < s \leq 0\,.\end{array}$$

In this manner we proceed; in general, we define the functions of our system in the following way: we divide the interval $[0,1]$ into 2^n equal parts and denote these subintervals in turn by $i_n^{(1)}, i_n^{(2)}, \cdots, i_n^{(2^n)}$. Now we put:

$$\begin{array}{rll}\chi_n^{(k)} = & 0 & \text{within the intervals} \quad i_n^{(1)}, i_n^{(2)}, \cdots, i_n^{(2k-2)}\,;\\ = & \sqrt{2^{n-1}} & \text{within the interval} \quad i_n^{(2k-1)}\,;\\ = & -\sqrt{2^{n-1}} & \text{within the interval} \quad i_n^{(2k)}\,;\\ = & 0 & \text{within the intervals} \quad i_n^{(2k+1)}, \cdots, i_n^{(2^n)}\\ & & (k = 1, 2, \cdots, 2^{n-1}).\end{array}$$

At the points 0 and 1 we assign to each function $\chi_n^{(k)}(s)$ being constant on the interval $\left[0, \frac{1}{2^n}\right]$ or $\left[1-\frac{1}{2^n}, 1\right]$, respectively, the value it assumes in these respective intervals. Thus $\chi_n^{(k)}(s)$ is a piecewise constant function which is continuous with the exception of the points $\frac{2k-2}{2^n}$, $\frac{2k-1}{2^n}$, $\frac{2k}{2^n}$, where it suffers a finite jump. We now make the agreement that at these points, $\chi_n^{(k)}$ be equal to the arithmetic means of the values it assumes in the intervals adjoining at that very point.

We now claim that the countably infinitely many functions

$$\chi: \qquad \chi_0(s),\ \chi_1(s),\ \chi_2^{(1)}(s), \chi_2^{(2)}(s),\ \chi_3^{(1)}(s),\ \chi_3^{(2)}(s),\ \cdots$$

so defined *form a complete orthogonal function system*. Indeed, if $\chi_n^{(k)}(s)$ and $\chi_\nu^{(\varkappa)}(s)$ are two different functions of the system χ and we have $n > \nu$, then $\chi_\nu^{(\varkappa)}(s)$

has a constant value on the entire interval where $\chi_n^{(k)}(s)$ is different from zero. Therefore, we have

$$\int_0^1 \chi_n^{(k)}(s)\,\chi_\nu^{(\varkappa)}(s)\,ds = \text{const.} \int_0^1 \chi_n^{(k)}(s)\,ds = 0\,,$$

from which we conclude immediately that the function system χ possesses the *orthogonality condition*. In order to show that also the *completeness relation* is satisfied, it obviously suffices to prove that any function $f(s)$ which is integrable in the Lebesgue sense and which for all pairs n, k coming into question satisfies the relation

$$\int_0^1 f(s)\,\chi_n^{(k)}(s)\,ds = 0\,, \tag{16}$$

vanishes identically up to a null set. For this purpose, let us consider the function

$$F(s) = \int_0^s f(s')\,ds'\,;$$

because of the equation

$$\int_0^1 f(s)\,\chi_0(s)\,ds = 0\,,$$

we have

$$F(1) = 0\,.$$

The last equation yields in connection with the equation

$$\int_0^1 f(s)\,\chi_1(s)\,ds = \int_0^{\frac{1}{2}} f(s)\,ds - \int_{\frac{1}{2}}^1 f(s)\,ds = 0$$

the statement that $F\left(\frac{1}{2}\right) = 0$. From this and from the equations

$$\int_0^1 f(s)\,\chi_2^{(1)}(s)\,ds = \sqrt{2}\left\{\int_0^{\frac{1}{4}} f(s)\,ds - \int_{\frac{1}{4}}^{\frac{1}{2}} f(s)\,ds\right\} = 0\,,$$

$$\int_0^1 f(s)\,\chi_2^{(2)}(s)\,ds = \sqrt{2}\left\{\int_{\frac{1}{2}}^{\frac{3}{4}} f(s)\,ds - \int_{\frac{3}{4}}^1 f(s)\,ds\right\} = 0$$

we deduce that we have

$$F\left(\frac{1}{4}\right) = F\left(\frac{3}{4}\right) = 0\,,$$

etc. We can conclude in this way that the function $F(s)$ always equals zero if s is a finite binary fraction of the form $\frac{1}{2^{p_1}} + \cdots + \frac{1}{2^{p_n}}$, and these points form an everywhere dense point set. As we know, $F(s)$ is a continuous function, though, and we have with the exception of a set of measure zero that

$$f(s) = \frac{d}{ds}\big(F(s)\big)\,.$$

We deduce from this that $F(s)$ vanishes identically on the entire interval, and that $f(s)$ — apart from a point set of measure zero — also is zero everywhere. Thus the completeness of the considered orthogonal system is also proven and it is shown that *each integrable function satisfying the relations* (16) *vanishes except for a null set.*

§ 2.

Expansions with Respect to the Orthogonal Function System χ.

We now get to the most important point of this investigation by showing that the *Fourier series of each continuous function on the interval* $[0, 1]$ *taken with respect to the orthogonal system* χ *defined just now converges uniformly to this function.*

Let us cut off the infinite series

$$\chi_0(s)\int_0^1 f(t)\,\chi_0(t)\,dt + \chi_1(s)\int_0^1 f(t)\,\chi_1(t)\,dt + \cdots$$

$$+\chi_n^{(1)}(s)\int_0^1 f(t)\,\chi_n^{(1)}(t)\,dt + \cdots + \chi_n^{(p)}(s)\int_0^1 f(t)\,\chi_n^{(p)}(t)\,dt + \cdots$$

at some term, at $\chi_n^{(p)}(s)\int_0^1 f(t)\,\chi_n^{(p)}(t)\,dt$, say; we obtain a finite sum, which we from now on denote by $[f(s)]_n^{(p)}$:

$$[f(s)]_n^{(p)} = \chi_0(s)\int_0^1 f(t)\,\chi_0(t)\,dt + \cdots + \chi_n^{(p)}(s)\int_0^1 f(t)\,\chi_n^{(p)}(t)\,dt\,.$$

If we set analogously as before

$$K_n^{(p)}(s,t) = \chi_0(s)\,\chi_0(t) + \cdots + \chi_n^{(1)}(s)\,\chi_n^{(1)}(t) + \cdots + \chi_n^{(p)}(s)\,\chi_n^{(p)}(t)\,,$$

then we have

$$[f(s)]_n^{(p)} = \int_0^1 K_n^{(p)}(s,t)\,f(t)\,dt\,.$$

The last equation defines infinitely many functions of two variables

$$K_0(s,t),\ K_1(s,t),\ K_2^{(1)}(s,t),\ K_2^{(2)}(s,t),\ \cdots$$

and we now turn to the investigation of the properties of these functions.

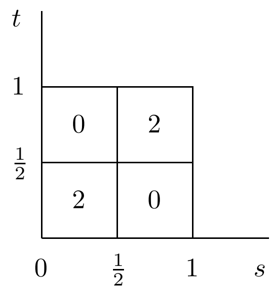
The function $K_0\,(s,t)$ defined in the square $0 \le s \le 1$, $0 \le t \le 1$ equals 1 everywhere. The function $\chi_1(s)\,\chi_1(t)$ equals 1 within the squares

$$Q_{11} : 0 \le s \le \tfrac{1}{2}\,, \quad 0 \le t \le \tfrac{1}{2} \qquad \text{and} \qquad Q_{22} : \tfrac{1}{2} \le s \le 1\,, \quad \tfrac{1}{2} \le t \le 1\,,$$

it equals -1 in the other two squares

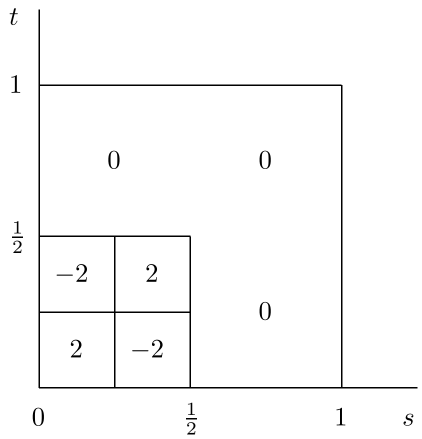
$$Q_{12} : \tfrac{1}{2} \le s \le 1\,, \quad 0 \le t \le \tfrac{1}{2} \qquad \text{and} \qquad Q_{21} : 0 \le s \le \tfrac{1}{2}\,, \quad \tfrac{1}{2} \le t \le 1\,;$$

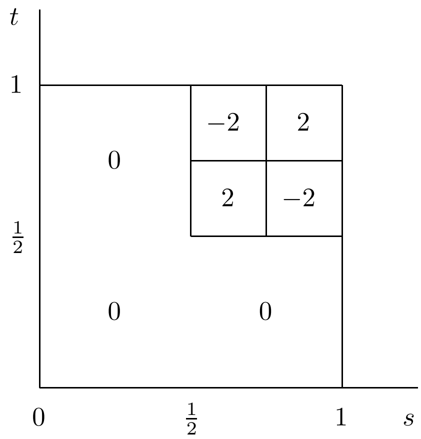
therefore, $K_1\,(s,t)$ equals 2 in Q_{11} and Q_{22}, but equals 0 in Q_{12}, Q_{21}.



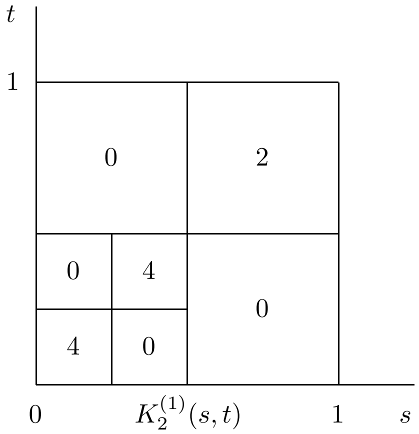
$K_1\,(s,t)$

At the lines $s = \frac{1}{2}$ and $t = \frac{1}{2}$, resp., the function $K_1\,(s,t)$ of course equals the arithmetic means of the values it assumes in the squares adjoining at that very place. In order to obtain furthermore $K_2^{(1)}(s,t)$ and $K_2^{(2)}(s,t)$, we indicate in the figure those values the functions $\chi_2^{(1)}(s)\,\chi_2^{(1)}(t)$ and $\chi_2^{(2)}(s)\,\chi_2^{(2)}(t)$ assume respectively.

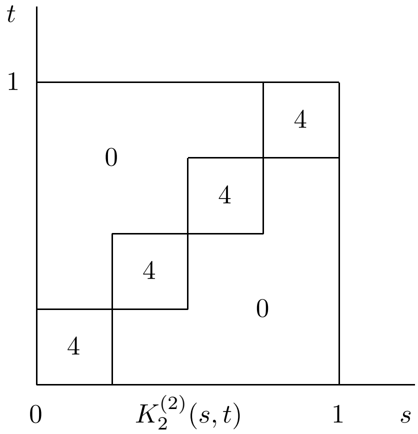




The values of $K_2^{(1)}(s,t)$ and $K_2^{(2)}(s,t)$ are thus represented graphically by the following figures.



$K_2^{(1)}(s,t)$



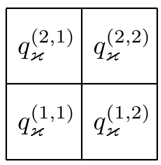
$K_2^{(2)}(s,t)$

From this, the formation priciple of the functions $K_n^{(p)}(s,t)$ is already apparent: To obtain the range of the function $K_n^{(2^n-1)}(s,t)$, we divide the unit square Q into 2^{2^n} equal subsquares; in the subsquares $q_1, \cdots, q_{2^n}$ lying along the diagonal $s=t$ of the square Q, we have

$$K_n^{(2^n-1)}(s,t) = 2^n\,;$$

within the other squares, $K_n^{(2^n-1)}(s,t)$ equals zero. At the points where this function suffers a jump, it assumes the arithmetic means of the values it has in the squares adjoining at that very place. To obtain now $K_{n+1}^{(p)}(s,t)$, say, we subdivide each of the first p subsquares $q_1, q_2, \cdots, q_p$ of the former division which lie along the diagonal into four equal squares, as the figure suggests:

$$q_\varkappa = q_\varkappa^{(1,1)} + q_\varkappa^{(1,2)} + q_\varkappa^{(2,1)} + q_\varkappa^{(2,2)} \qquad (\varkappa = 1, \cdots, p).$$



$q_\varkappa$

Then $K_{n+1}^{(p)}(s,t)$ is given by the following rule: In the subsquares $q_\varkappa^{(1,1)}$, $q_\varkappa^{(2,2)}$, we have $K_{n+1}^{(p)}(s,t) = 2^{n+1}$; in the subsquares $q_\varkappa^{(1,2)}$, $q_\varkappa^{(2,1)}$, though, it equals zero. Within all the other squares, we have $K_{n+1}^{(p)}(s,t) = K_n^{(2^n-1)}(s,t)$ *). At the points where $K_{n+1}^{(p)}(s,t)$ becomes discontinuous (*i.e.*, at those points s, t where one of the quantities s and t is a finite binary fraction of the form $\frac{1}{2^{p_1}} + \cdots + \frac{1}{2^{n+1}}$), we determine it according to the rule mentioned above.

*) *i.e.*, it equals 2^n in the subsquares $q_{p+1}, \cdots, q_{2^n}$, and equals 0, if the point s, t does not lie within one of these squares.

In order to prove the correctness of this rule, we assume it to be correct for $K_n^{(2^n-1)}(s,t)$; now we have

$$K_{n+1}^{(1)}(s,t) = K_n^{(2^n-1)}(s,t) + \chi_{n+1}^{(1)}(s)\,\chi_{n+1}^{(1)}(t)\,.$$

Since $\chi_{n+1}^{(1)}(s)$ differs from zero only in the interval $0 \leq s \leq \frac{1}{2^n}$, though, $K_{n+1}^{(1)}(s,t)$ can only differ from $K_n^{(2^n-1)}(s,t)$ in the square

$$0 \leq s \leq \frac{1}{2^n}\,, \quad 0 \leq t \leq \frac{1}{2^n}\,,$$

i.e., in q_1. Since we have

$$\begin{aligned} \chi_{n+1}^{(1)}(s)\,\chi_{n+1}^{(1)}(t) &= 2^n \quad \text{in the subsquares } q_1^{(1,1)},\ q_1^{(2,2)}, \\ &= -2^n \quad \text{in the subsquares } q_1^{(1,2)},\ q_1^{(2,1)}, \end{aligned}$$

though, it follows that $K_{n+1}^{(1)}(s,t)$ has the value just given:

$$\begin{aligned} K_{n+1}^{(1)}(s,t) &= 2^{n+1} \quad \text{in } q_1^{(1,1)},\ q_1^{(2,2)} \\ &= 0 \qquad\ \ \text{in } q_1^{(1,2)},\ q_1^{(2,1)}. \end{aligned}$$

Let us denote by $f(s)$ an arbitary function, integrable in the Lebesgue sense and defined in the interval $[0,1]$, and by $s = a$ an arbitrary point of the interval. Then we have

$$[f(a)]_n^{(p)} = \int_0^1 K_n^{(p)}(a,t)\,f(t)\,dt\,;$$

if we assume for the moment that a be no finite binary fraction of the above form, then the function $K_n^{(p)}(a,t)$ of t equals zero everywhere except for an interval $i_n^{(p)}$ whose length $l_n^{(p)}$ equals $\frac{1}{2^{n-1}}$ or $\frac{1}{2^n}$. In this interval $i_n^{(p)}$, though, we have $K_n^{(p)}(a,t) = \frac{1}{l_n^{(p)}}$ and we thus find that

$$[f(a)]_n^{(p)} = \frac{1}{l_n^{(p)}} \int\limits_{(i_n^{(p)})} f(t)\,dt\,.$$

Is, however, a a finite binary fraction, then $K_n^{(p)}(a,t)$ is different from zero in an interval $\bar{i}_n^{(p)}$ whose length is

$$\bar{l}_n^{(p)} = \frac{1}{2^{n-2}} \quad \text{or} \quad = \frac{1}{2^{n-1}}\,;$$

the value of $K_n^{(p)}(a,t)$ in this interval equals $\frac{1}{\bar{l}_n^{(p)}}$, though, and thus we obtain in this case also

$$[f(a)]_n^{(p)} = \frac{1}{\bar{l}_n^{(p)}} \int\limits_{(\bar{i}_n^{(p)})} f(t)\,dt\,.$$

In both cases the length of the integration interval — which contains the point $t = a$ — thus equals the reciprocal value of the factor standing in front of the interval.

But now $l_n^{(p)}$ and $\bar{l}_n^{(p)}$ converge to zero with growing n, and therefore the partial sums $[f(a)]_n^{(p)}$ converge to

$$\underset{n=\infty}{L}\frac{1}{l_n^{(p)}}\int\limits_{(i_n^{(p)})} f(t)\,dt\,.$$

Since the intervals $i_n^{(p)}$ shrink to the point $t=a$ with growing n, this limit is nothing else but the value of the differential quotient of $\int\limits_0^s f(t)\,dt$ with respect to s at the point $s=a$:

$$\underset{n=\infty}{L}\frac{1}{l_n^{(p)}}\int\limits_{(i_n^{(p)})} f(t)\,dt = \left[\frac{d}{ds}\left(\int\limits_0^s f(t)\,dt\right)\right]_{s=a},$$

and we have the result: *Is $f(s)$ an arbitrary function, then the partial sums of its expansion converge at every point $s=a$ where the differential quotient $\frac{d}{ds}\left(\int\limits_0^s f(t)\,dt\right)$ exists, and represent this value.*

Since, however, — by a theorem of Lebesgue — everywhere except for a set of measure zero,

$$\frac{d}{ds}\left(\int\limits_0^s f(t)\,dt\right)$$

exists and agrees with $f(s)$, this implies: *the expansion of an arbitrary function with respect to the functions of our orthogonal system converges at every point with the exception of a point set of measure zero.*

Is $f(s)$ continuous at every point of the interval, though, then we have, as is well known,

$$f(s)=\frac{d}{ds}\left(\int\limits_0^s f(t)\,dt\right)$$

for every point without exception, *i.e.*, *the Fourier expansion* (*with respect to our orthogonal system* χ) *of an arbitrary continuous function converges at every point of the interval* $[0,1]$. *)

§ 3.

Further Properties of the Orthogonal Function System χ.

Now we want to derive some further properties of our function system, corresponding to those theorems in the theory of trigonometric series obtained there from various summation methods.

*) In a note that appeared recently in the "Jahresberichte der deutschen Mathematiker-Vereinigung" (1910), Mr. Faber has made my orthogonal system the object of his investigations and derives my theorems afresh. His method of proof is not essentially different from the one given here, though.

From the circumstance that the functions $K_n^{(p)}(s,t)$ are always positive we conclude the following theorem:

If the function $f(s)$, integrable in the Lebesgue sense in the interval $[0,1]$, remains between the bounds m and M:

$$m \le f(s) \le M\,,$$

then all partial sums of the Fourier series of $f(s)$ with respect to χ also remain between these bounds:

$$m \le [f(s)]_n^{(p)} \le M\,.$$

Indeed, we have, *e.g.*,

$$[f(s)]_n^{(p)} = \int\limits_0^1 K_n^{(p)}(s,t)\,f(t)\,dt \le M \int\limits_0^1 K_n^{(p)}(s,t)\,dt = M\,.$$

Let now $s = a$ be an arbitrary point of the interval $[0,1]$. Since for sufficiently large n, the functions $K_n^{(p)}(a,t)$ will certainly vanish if

$$0 \le t \le a - \varepsilon$$

or if

$$a + \varepsilon \le t \le 1\,,$$

however small the positive number ε has been chosen, we have

$$[f(a)]_n^{(p)} = \int\limits_{a-\varepsilon}^{a+\varepsilon} K_n^{(p)}(a,t)\,f(t)\,dt\,,$$

whenever n exceeds a certain bound. Since in this formula, the behaviour of the function $f(s)$ in the intervals $0 \le s \le a-\varepsilon$ and $a+\varepsilon \le s \le 1$, resp., is not being expressed at all, we conclude from this that *the convergence of the Fourier series of an arbitrary function taken with respect to χ at the point $s = a$ only depends on the behaviour of this function in the neighbourbood of this point.*

If the functions $f(s)$ and $g(s)$ agree in however small an interval, then we can specify an index N such that whenever we have $n > N$, all the $[f(s)]_n^{(p)}$ and $[g(s)]_n^{(p)}$ also agree in this interval.

In connection with the main theorem of the paragraph before, these latter results yield the following theorem: *the Fourier series with respect to χ of a function $f(s)$ converges to this function at every point of continuity of $f(s)$.*

§ 4.

Various Generalizations.

We can generalize the orthogonal system just defined in various directions without destroying its essential properties. We now want to outline some of these generalizations.

To begin with, we can use a more general method in dividing up the s-axis for the construction of a similar orthogonal system in the following way: the first function of the system again be equal to 1; then we choose an arbitrary point in the interval $[0,1]$, say, α_1, and construct a function $\overline{\chi}_1(s)$ which is constant on the intervals $[0,\alpha_1]$ and $[\alpha_1,1]$, respectively, and furthermore satisfies the two conditions

$$\int_0^1 \overline{\chi}_1(s)\,ds = 0\,, \qquad \int_0^1 \big(\overline{\chi}_1(s)\big)^2\,ds = 1\,.$$

Then we choose two points $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$, respectively, arbitrarily in the intervals $[0,\alpha_1]$ and $[\alpha_1,1]$, and determine the functions $\overline{\chi}_2^{(1)}(s)$ and $\overline{\chi}_2^{(2)}(s)$ according to the following rule: $\overline{\chi}_2^{(1)}(s)$ shall vanish on $[\alpha_1,1]$ and assume on the intervals $[0,\alpha_2^{(1)}]$ and $[\alpha_2^{(1)},\alpha_1]$ a constant value each which shall be chosen in such a way that we have

$$\int_0^1 \overline{\chi}_2^{(1)}(s)\,ds = 0\,, \qquad \int_0^1 \big(\overline{\chi}_2^{(1)}(s)\big)^2\,ds = 1\,.$$

$\overline{\chi}_2^{(2)}(s)$, though, shall vanish on $[0,\alpha_1]$ and assume on the intervals $[\alpha_1,\alpha_2^{(2)}]$ and $[\alpha_2^{(2)},0]$ a constant value each which shall be chosen in such a way that the relations

$$\int_0^1 \overline{\chi}_2^{(2)}(s)\,ds = 0\,, \qquad \int_0^1 \big(\overline{\chi}_2^{(2)}(s)\big)^2\,ds = 1$$

are satisfied. Now we choose four points $\alpha_3^{(1)}$, $\alpha_3^{(2)}$, $\alpha_3^{(3)}$, $\alpha_3^{(4)}$, which shall lie in the intervals $[0,\alpha_2^{(1)}]$, $\cdots$, $[\alpha_2^{(2)},1]$, respectively, and construct in corresponding manner the functions $\overline{\chi}_3^{(1)}(s)$, $\cdots$, $\overline{\chi}_3^{(4)}(s)$; and we proceed in this way. Again we make the agreement that at the points of discontinuity, the functions be equal to the arithmetic means of the values they assume in the intervals meeting at that very place.

If the points $\alpha_n^{(p)}$ chosen in this manner form an everywhere dense point set, then we can conclude in the exactly same way as earlier on p. 28 that the *orthogonal system so defined is complete.* In order to show that this function system also has the property that all continuous functions can be expanded in the Fourier manner into a series progressing according to the functions of this system, we note that the functions

$$\overline{K}_n^{(p)}(s,t) = \overline{\chi}_0(s)\,\overline{\chi}_0(t) + \cdots + \overline{\chi}_n^{(1)}(s)\,\overline{\chi}_n^{(1)}(t) + \cdots + \overline{\chi}_n^{(p)}(s)\,\overline{\chi}_n^{(p)}(t)$$

remain positive for every pair of values n, p and that we have

$$\int_0^1 \overline{K}_n^{(p)}(s,t)\,dt = 1\,.$$

If we now by $[f(s)]_n^{(p)}$ again denote the finite sum we obtain by terminating the Fourier series of the function $f(s)$ at the term $\overline{\chi}_n^{(p)}(s)\int\limits_0^1 f(t)\,\overline{\chi}_n^{(p)}(t)\,dt$, then we have

$$(17) \qquad [f(s)]_n^{(p)} = \int\limits_0^1 \overline{K}_n^{(p)}(s,t)\,f(t)\,dt\,;$$

and since the functions $\overline{K}_n^{(p)}(s,t)$ are always positive, we may conclude that $[f(s)]_n^{(p)}$ always remains between the maximum and the minimum of $f(s)$. In other words, the assignment given by dint of equation (17) satisfies all conditions of our lemma p. 18 which implies that the *Fourier series with respect to the orthogonal system under consideration of an arbitrary function $f(s)$ converges uniformly to this function, if $f(s)$ lies in the range of the orthogonal system.*

Now, however, it is immediately clear that every finite aggregate of our orthogonal functions is a piecewise constant function; conversely, every piecewise constant function which suffers a jump only at finitely many points $\alpha_n^{(p)}$ and at such a point equals the arithmetic means of the values it assumes in the intervals meeting at that very place, is a finite aggregate of our orthogonal functions. Since, however, the points of discontinuity $\alpha_n^{(p)}$ are distributed everywhere dense in the interval $[0,1]$, we see immediately that every continuous function can be approximated arbitrarily by such a piecewise constant function. Thus it is shown that *the range of our orthogonal system comprises all continuous functions, and thus the Fourier series of any continuous function converges uniformly in the whole interval.* It also does not cause any further difficulties to furnish the proof that the series of any integrable function is convergent everywhere with the exception of a null set.

A further generalization of our orthogonal system could be obtained by performing, instead of the bisection of the intervals, a trisection, or quadrusection, *etc.*; then we define, in the same manner as before, a complete orthogonal function system by constructing at each subdivision of the intervals a finite system of piecewise constant functions which are orthogonal to each preceding function, and which in the divided intervals assume a constant value each. If we define — which after all is always possible — at *each* subdivision such a number of functions that there exists no piecewise constant function not vanishing identically, which is orthogonal to all functions defined *up to now*, and which only suffers a finite jump at the division points of the *current* subdivision; then — if the subdivision points form an everywhere dense point set — the function systems thus obtained are complete in that general sense that every Lebesgue integrable function which is orthogonal to all functions of the system vanishes except for a set of measure zero. They all own the convergence property specified above, and finally, the theorem also holds that the convergence of the Fourier series (with respect to these function systems) of a function at a point only depends on the behaviour of the function in the neighbourhood of this point.