WEIERSTRASS' THEOREM BEFORE WEIERSTRASS

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Abstract. Some remarks on the development of ideas leading to the Weierstrass approximation theorem are given. They are meant as comments on the Pinkus article [Pi] only. Hence the references containing numbers are related to the bibliography in [Pi] while references with letters can be found at the end of this remark.

Introduction. The recent Pinkus article [Pi] sheds light on the development of proofs of the *Weierstrass approximation theorem.* It contains a careful exposition of different contributions to this process based on a detailed study of almost all relevant sources. I would like to add a few comments mostly concerning the time when relevant notions were not sufficiently ripe and clear but nevertheless were used by many who knew, more or less correctly, how to handle them.

Let us start at the beginning of the nineteenth century. Some rather deep theorems such as the fundamental theorem of algebra had already been proved but notions such as the concept of function, its continuity, etc., were by no means yet clear. So the assertion of the possibility of expressing any (continuous) function as a power series as used by Joseph L. Lagrange (1736 – 1813) [La] was true when a function meant nothing but the corresponding analytic expression. However, functions were also defined by (possibly divergent) series or as solutions of equations, and even first-class mathematicians used unjustified operations or methods of proofs.

Both Bernard Bolzano (1781 – 1848) and Augustin L. Cauchy (1789 – 1857) lived during that time. Let us take as an example Bolzano's proof (*Rein analytischer Beweis* . . . , 1817) of the intermediate value property of continuous functions. Despite his outstanding mathematical talent Bolzano provided a proof on the level of mathematical standards of that time. Note also that Bolzano's attempt to introduce real numbers was far from acceptable by present standards.

Notions of uniform convergence and uniform continuity were introduced later. The introduction of the concept of uniform convergence is connected with several names but the most important among them is that of Karl T. W. Weierstrass (1815 – 1897). Weierstrass learned this concept from his teacher Christof Gudermann (1798 – 1852). He mastered it and used it properly. The importance of

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these notions was soon recognized. Bolzano was already dead but Cauchy was still alive.

In 1853 Cauchy succeeded in correcting his previous assertion from [Ca] (1820) on the continuity of pointwise convergent series of continuous functions.

Nowhere differentiable continuous functions. Bolzano's construction of a continuous nowhere differentiable continuous (NDC) function differs from the one used by Weierstrass. It is based on a "geometric description" of a sequence of piecewise linear functions converging to a function with the required property. This sequence is uniformly convergent but needless to say Bolzano, like Cauchy, made the usual error of asserting the continuity of the limit function from the pointwise convergence of continuous functions of the sequence. A detailed study of the content of Bolzano's *Functionenlehre* was carried out by Vojtěch Jarník (1897 – 1970). He remarked that Bolzano's text on the non-differentiable function was written during the period of 1831-34.

Bolzano was admirably accurate in many respects (e.g. in defining continuity and other important notions), but he was not a professional mathematician; in fact, he was a professor of theology. We can accept that he proved the non-differentiability of "his function" only on a dense subset of an interval. His ideas were completed into a definitive proof much later when the manuscript of *Functionenlehre* was discovered in Vienna by Martin Jašek (1879 – 1945) soon after the World War I[1]. Note that some articles in Czech on NDC functions appeared before this discovery (e.g. Čupr [Cu] (1912), Petr [Pe1] (1920)).

Neither Bolzano's example nor the example of Charles Cellérier (1818 – 1889) (approx. around 1860) had a chance to influence the development of relevant ideas, since they remained completely unknown to contemporary mathematicians. To complete the picture let us quote from Weierstrass' work [107] a bit more than is usual (after the sign • it is almost the exact quotation of Bottazzini's translation; see [Bo], p. 249):

Only recently it was generally believed that a continuous function of a real variable has a derivative the value of which is not defined or is infinite only at several isolated points. Even in the works of Gauss, Cauchy or Dirichlet, mathematicians who were rather critical of everything in their fields, we are unable — as far as I know — to find a mark of a different opinion. I learned from Riemann's students that he was the first who expressed a conviction (in 1861 or even sooner) that this assertion

[1] Martin Jašek was a secondary school teacher. He studied philosophy and mathematics and his thesis was in philosophy (written under Prof. Thomas G. Masaryk (1850 – 1937) who later became the first President of Czechoslovakia). It is not precisely known when Jašek discovered Bolzano's function (1919 ?) but his lecture (16.12.1921) was a sensation and some other mathematicians published proofs of non-differentiability of the function. Jarník's article on the subject appeared in the same year as Jašek's article on Bolzano's function (1922). Other related works were written e.g. by Gerhard Kowalewski (1876 – 1950) (1922, 1923), Karel Petr (1868 – 1950) (1923) and Karel Rychlík (1885 – 1968) (1922).

is not valid, e.g., for a function expressed by infinite series

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}\,.$$

• *Unfortunately* [Weierstrass continues] *Riemann's proof of this has not been published, nor does it appear to have been preserved in his papers or by oral communication. It is more regrettable that I had not learned for sure what Riemann exactly said about the example. Those mathematicians who have concerned themselves with the problem after Riemann's conjecture became known in wider circles, seem to be of the opinion (at least a majority thereof) that it is enough to prove the existence of functions that have points in every interval of their argument, no matter how small, where they are not differentiable. That there are functions of this kind is unusually easy to prove, and I consequently believe that Riemann had in mind only those functions that have no determinate derivative for any value of their argument.* (See also [We], pp. 71–72.)

A reader more interested in the problematic of NDC functions should consult works devoted entirely to the problem (cf. for example [Bo], [Sin], [Vo]). Here we list only the most important facts:

(a) A more systematic study of "pathological functions" was started by Bernhard G. F. Riemann (1826 – 1866) in 1854; cf. [Ri2].

(b) The first example of a NDC function was given by Weierstrass in 1872 (published in 1875 by his pupil Paul Du Bois-Reymond (1831 – 1889)).

(c) Previously existing examples of NDC functions (Bolzano, Cellérier, Riemann) had an indirect influence on the further development and were fully understood only much later. It turns out that only Bolzano's function is NDC while derivatives of the functions considered by Riemann and Cellérier exist on a dense subset of an interval.

It is very likely that Weierstrass in 1872 believed that Riemann's function is NDC (see the quotation above). Now we know what it took a long time to decide, that this is not true. This Weierstrass belief could have been a reason for the late publication of his result. On the other hand there is little doubt that his example of an NDC function (or any other one) was known to him since 1861.

In our opinion, an example of an NDC function should be presented in any course on Real Analysis in connection with the concept of uniform convergence. Weierstrass' example of an NDC function is still somewhat technically difficult but a modified example of Teiji Takagi (1875 – 1960) (1903) as well as a similar example now mainly called the *Van der Waerden's example* are available[2]; cf. [86] or [St].

Approximation theorem. Shortly before 1885 when Weierstrass started to work on the approximation theorem he was not only rather old but also very depressed.

[2] In fact Van der Waerden's example is based on a solution of a problem which was solved by Heyting and Buseman.

He even considered leaving Berlin because of personal problems with Leopold Kronecker (1823 – 1891). During that time Sophia Kowalewski (1850 – 1891), Weierstrass' favorite student, discussed with him problems concerning her lectures on PDEs. It should be mentioned that the singular integral providing the solution of Cauchy's problem for the half-space and the heat equation is usually associated with Weierstrass' name. This was the source of inspiration for his proof. Weierstrass himself followed different goals than uniform approximation (Euler's definition of a function as an "analytic expression", cf. [SS]), but recognized the importance of the result obtained.

The related problem on the heat equation can be viewed from the point of view of potential theory as the *Dirichlet problem* (DP). Here we are close to another indirect influence of Riemann on approximation problems. As mentioned in [Pi] (see p. 22), the proof of Charles E. Picard (1856 – 1941) was inspired by the article [84] of another Weierstrass pupil, Hermann C. A. Schwarz (1843 – 1921)[3]. During the years 1870–1872, Schwarz published several articles on the DP for harmonic functions (in fact, he was the first to prove, in 1870, that the *Poisson integral* gives the solution of the DP for the ball; in [84] he was quite close to the proof of Weierstrass' theorem). His motivation was the general interest of mathematicians in placing Riemann's results in complex function theory (cf. [Ri1]) on a solid basis after the criticism of Weierstrass. Needless to say, this relationship between the problem of uniform approximation and the Dirichlet problem was not arbitrary; compare with another article [102] on the further development of proofs of the Weierstrass theorem.

On the other hand, the Weierstrass theorem in several variables turned out to be a powerful tool which was widely used in potential theory. One of the first applications of Weierstrass' theorem is also from this field: already in 1887 Henri Poincaré (1854 – 1912) announced his solution of the Dirichlet problem by the so called *méthode de balayage* (see [Po] or the classical book [Ke]). Thus, the theorem started to repay potential theory for the initial hint for its proof.

It was in the air. As known from [Pi], another Weierstrass pupil Carl D. T. Runge (1856 – 1927) almost proved this theorem at about the same time. But he did not formulate it explicitly and the result appeared as an observation in a letter from Lars E. Phragmén (1863 – 1937) to Gösta Mittag-Leffler (1846 – 1927) in 1886; see [64]. In the previous section we mentioned that Schwarz too was very close to the Weierstrass theorem but he also did not formulate it. Were these two mathematicians the only people who could have easily obtained the theorem before Weierstrass?

The answer is in the negative. Let us consider the different proofs from a particular point of view. We can easily find other differences among the proofs of the Weierstrass theorem. The first known proofs can be described as "proofs in two steps": a continuous function is first approximated by a function (analytic or piecewise linear) and then the latter function by a polynomial. Later, "one-

[3]Under the influence of Weierstrass and Kummer, Schwarz switched from the study of Chemistry to Mathematics. Later he learned much from both of these men.

step proofs" were published based on the direct construction of the approximating polynomial. As examples we mention the proof of Edmund G. H. Landau (1877 – 1938) from 1908 (see [52]) and the proof of Sergey Natanovich Bernstein (1880 – 1968) from 1912 (see [5]). Similar proofs in terms of trigonometric polynomials were obtained by Charles de la Vallée Poussin (1866 – 1962); cf. [Pi], p. 41–42 (see [98]). At first glance, the Bernstein polynomials are the simplest since the corresponding formula is "discrete" but there are also discrete analogues of Landau's and de la Vallée Poussin's integrals; cf. [Si] and [Kr]. It was probably easier to discover a two-step proof and hence we should analyze this approach.

An idea of the two-step proof is mentioned in [Pi] in connection with the proofs of Mathias Lerch (1860 – 1922) and Vito Volterra (1860 – 1940). They both, independently of each other, approximated the function in question by a polygonal line, and then developed this function (which is obviously continuous and piecewise monotone) in its Fourier series. Since the time of the discovery by Pierre G. L. Dirichlet (1805 – 1859) it was known that the Fourier series of such a function is *pointwise* convergent. Many famous mathematicians were interested in the field of Fourier series and so many of them had the same chance. Obviously, it remained to prove that in such a case — and provided the function is also 2π-periodic — the series is *uniformly* convergent on $\mathbb{R}$. However, this had already been proved in 1870 by Eduard H. Heine (1821 – 1881); see [He]. Thus to make the observation and connect two (from the present point of view) simple facts should have been easy; but it was not easy *at that time*.

Allow me to add a few comments about Lerch who was mentioned above. At the end of the nineteenth century he was probably the most talented Czech mathematician. He wrote 238 articles but he did not obtain a position in Prague and hence spent a long time abroad (1896 – 1906, Fribourg, Switzerland). His results were very much appreciated by Charles Hermite (1822 – 1901). In 1906 he was appointed professor of mathematics at the Czech Technical University in Brno where — many years later — Jarník became his assistant.

Lerch was one of those who learned the ideas of uniform convergence directly from Weierstrass while studying in Berlin (1884/85). He had come to Berlin from Prague to learn more from Weierstrass, but he became a pupil of Kronecker. However what he learned from Weierstrass was not lost. It helped him throughout his professional career. From the point of analysis it might be of interest to recall that he constructed a relatively simple example of an infinitely smooth function on $\mathbb{R}$

$$f(x) = \sum_{k=0}^{\infty} \frac{\cos(a^n x)}{n!}$$

(a odd and > 1) which is not analytic at any point; see [Le]. He also contributed to questions concerning domains of holomorphy ("natural boundaries"). In his almost unknown article [56] (in Czech) he mentioned also Runge's lectures in 1885 which included an implicit proof of Weierstrass' theorem.

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