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On Approximate Polynomials,

By

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Lately, Mr. J. Pál has proved the following interesting theorem : [1] Let $f(x)$ be a continuous function of a real variable x in the interval $0 \leqq |x| \leqq a < 1$, which vanishes at the point $x=0$, and let ε be an arbitrary positive number, then there exists a polynomial $P(x)$ with integral coefficients such that

$$|f(x)-P(x)| < \varepsilon,$$

for all values of x in the interval $0 \leqq |x| \leqq a$.

In his theorem, it is necessary that the number a, which is the upper limit of $|x|$, is less than unity. To extend the theorem to the case when a is equal to unity is the aim of the following lines.

1. For our purpose, it is necessary to introduce a certain new condition for the given function $f(x)$; and the theorem thus extended runs as follows:

Let a function $f(x)$ be continuous in the interval $0 \leqq |x| \leqq 1$ and

$$f(0)=f(1)=f(-1)=0,$$

then, for any given positive number ε, there exists a corresponding polynomial $P(x)$ with integral coefficients such that

$$|f(x)-P(x)| < \varepsilon$$

for all values of x in the interval $0 \leqq |x| \leqq 1$.

To prove this theorem, we first consider an auxiliary polynomial

$$y = x(x+1)(x-1). \qquad (1)$$

As it is easily seen, the new variable y varies monotonously from 0 to $\frac{2}{3\sqrt{3}}$, while x varies from -1 to $-\frac{1}{\sqrt{3}}$, and y varies monotonously from $\frac{2}{3\sqrt{3}}$ to 0, while x varies from $-\frac{1}{\sqrt{3}}$ to 0. Consequently the two values p and q of x such that

(1) Tôhoku Math. Jour. vol. 6, 1914, p. 42.

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vi) $\varphi(x)=$
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function of y

$$y(-1+q)=y(-p), \tag{2}$$

$$0 \leqq q \leqq 1-\frac{1}{\sqrt{3}}, \qquad 0 \leqq p \leqq \frac{1}{\sqrt{3}}, \tag{3}$$

correspond one to one, and any one of them vanishes when the other vanishes. When x varies in the interval $(0, 1)$, the variation of y is symmetric with respect to the former, only the signs being different. Consequently we must have

$$y(1-q)=y(p). \tag{4}$$

Therefore, if a function $\varphi(x)$ is continuous in the interval $(-1, 1)$ and has the properties

$$\varphi(-1+q)=\varphi(-p), \quad \varphi(1-q)=\varphi(p), \tag{5}$$

then $\varphi(x)$ is evidently a uniform continuous function of y in the interval

$$\left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right). \tag{6}$$

Let it be denoted by

$$\varphi(x)=\psi(y). \tag{7}$$

Since $\varphi(x)$ is arbitrary in the interval $0 \leqq |x| \leqq \frac{1}{\sqrt{3}}$, we can define it in the following manner.

Take any two values p_1 and $p_2\,(>p_1)$ of p and corresponding two values q_1 and q_2 of q, and let

i) $\varphi(x)=0$ for $0 \leqq |x| \leqq p_1$,

ii) $\varphi(x)$ varies linearly from 0 to 1 in the intervals $p_1 \leqq |x| \leqq p_2$,

iii) $\varphi(x)=1$ for $p_2 \leqq |x| \leqq \frac{1}{\sqrt{3}}$.

Then the form of $\varphi(x)$ in the remaining intervals is necessarily determined as follows:

iv) $\varphi(x)=1$ for $\frac{1}{\sqrt{3}} \leqq |x| \leqq 1-q_2$,

v) $\varphi(x)$ varies monotonously from 1 to 0 in the intervals $1-q_2 \leqq |x| \leqq 1-q_1$,

vi) $\varphi(x)=0$ for $1-q_1 \leqq |x| \leqq 1$.

According to this definition of $\varphi(x)$, $\psi(y)$ is a uniform continuous function of y in the interval $0 \leqq |y| \leqq \frac{2}{3\sqrt{3}}<1$ and vanishes at the point $y=0$. Consequently, by the theorem of Mr. Pál, we can find a polynomial $Q(y)$ with integral coefficients such that

$$|\psi(y)-Q(y)|<\varepsilon_1 \quad \text{for} \quad 0 \leqq |y| \leqq \frac{2}{3\sqrt{3}}. \tag{8}$$

If we put

$$Q(y)=R(x),$$

$R(x)$ is also a polynomial with integral coefficients and is such that

$$|\varphi(x)-R(x)|<\varepsilon_1 \quad \text{for} \quad 0 \leqq |x| \leqq 1. \tag{10}$$

Again, by the same theorem, we can find a polynomial $S(x)$ with integral coefficients such that

$$|f(x)-S(x)|<\varepsilon_2 \quad \text{for} \quad 0 \leqq |x| \leqq 1-q_1. \tag{11}$$

From (10) and (11), we get

$$\begin{aligned} |f(x)\,\varphi(x)-S(x)\,R(x)| &< |f(x)|\,\varepsilon_1+|\varphi(x)|\,\varepsilon_2+\varepsilon_1\varepsilon_2 \\ &< M\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2, \end{aligned} \tag{12}$$

for the interval $0 \leqq |x| \leqq 1-q_1$, where M is the greatest magnitude of $|f(x)|$ in the interval $(-1, 1)$. Specially, if we consider only the interval in which $\varphi(x)$ becomes 1, we get

$$|f(x)-S(x)\,R(x)|<M\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2 \quad \text{for} \quad p_2 \leqq |x| \leqq 1-q_2. \tag{13}$$

Since, in the intervals $p_1 \leqq |x| \leqq p_2$ and $1-q_2 \leqq |x| \leqq 1-q_1$, $\varphi(x)$ varies monotonously from 1 to 0, we have

$$\begin{aligned} |f(x)-S(x)\,R(x)| &\leqq |f(x)\,\varphi(x)-S(x)\,R(x)|+|f(x)-f(x)\,\varphi(x)| \\ &< M(p_1, p_2)+M\varepsilon_1+\varepsilon_2+\varepsilon_1\varepsilon_2 \end{aligned} \tag{14}$$

$$\text{for} \quad p_1 \leqq |x| \leqq p_2 \quad \text{or} \quad 1-q_2 \leqq |x| \leqq 1-q_1,$$

where $M(p_1, p_2)$ is the greatest magnitude of $|f(x)|$ in the intervals of (14).

In the remaining intervals $\varphi(x)$ becomes zero and hence $|R(x)|$ becomes less than ε_1, so we have

$$\begin{aligned} |f(x)-S(x)\,R(x)| &\leqq |f(x)|+|S(x)|\,|R(x)| \\ &< M(p_1)+N(p_1)\,\varepsilon_1 \end{aligned} \tag{15}$$

$$\text{for} \quad 0 \leqq |x| \leqq p_1 \quad \text{or} \quad 1-q_1 \leqq |x| \leqq 1,$$

where $M(p_1)$ and $N(p_1)$ are the greatest magnitudes of $|f(x)|$ and $|S(x)|$ respectively in the intervals of (15).

Take p_1 and p_2 in the above discussion sufficiently small, then $M(p_1, p_2)$ and $M(p_1)$ become sufficiently small, for $f(x)$ is a continuous function vanishing at the points 0, 1 and -1. Next take ε_2 sufficiently small, then the quantity $N(p_1)$ is determined. Lastly, take ε_1 so small that $M\varepsilon_1$ and $N(p_1)\,\varepsilon_1$ also become sufficiently small. Then the right hand members of all the inequalities (13), (14) and (15) become sufficiently small. Hence, combining those three inequalities, we get

$$|f(x)-S(x)\,R(x)|<\varepsilon \tag{16}$$

for all values of x in the combined interval $0\leqq|x|\leqq 1$, where ε can be supposed to be an arbitrarily small number.

If we put

$$S(x)\,R(x)=P(x), \tag{17}$$

$P(x)$ is also a polynomial with integral coefficients, and, from (16) we get

$$|f(x)-P(x)|<\varepsilon \tag{18}$$

for all values of x in the interval $0\leqq|x|\leqq 1$. Thus our theorem is proved.

2. In the preceding theorem, we have given the condition that $f(x)$ vanishes at the points 0, 1 and -1. This condition can be replaced by the condition that $f(x)$ *takes such the integral values at the points* 0, 1 *and* -1 *that* $f(1)+f(-1)$ *is even.* For, in such a case, the function

$$g(x)=f(x)-\left[f(0)+\frac{f(1)-f(-1)}{2}\,x+\frac{f(1)+f(-1)-2f(0)}{2}\,x^2\right]$$

vanishes at the said three points and $g(x)-f(x)$ is a polynomial with integral coefficients.

The above new condition is also necessary. For, since $f(0)$, $f(1)$, $f(-1)$ can be approached indefinitely near by the integers $P(0)$, $P(1)$, $P(-1)$ respectively, they must be also integers and

$$f(0)=P(0),\quad f(1)=P(1),\quad f(-1)=P(-1),$$

for sufficiently small ε. That

$$f(1)+f(-1)=P(1)+P(-1)$$

must be even is a special consequence of the following general theorem: (¹)

The necessary and sufficient condition that the integral values u_1, u_2,

(¹) This follows at once from Newton's formula of interpolation.

...., u_n can be attained by a polynomial $P(x)$ with integral coefficients, for the integral values $a_1, a_2, \ldots, a_n$ of x, is that all of the $n-1$ expressions

$$\frac{u_1}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_k)}+\frac{u_2}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_k)}+\cdots$$
$$+\frac{u_k}{(a_k-a_1)(a_k-a_2)\ldots(a_k-a_{k-1})} \qquad k=2,\ 3,\ldots,\ n$$

should be integers.

To extend the theorem to an interval greater than or equal to $(-2,\ 2)$ is impossible, unless the function $f(x)$ itself is a polynomial in that interval. For if there are two different polynomials $P_1(x)$ and $P_2(x)$ with integral coefficients such that

$$|f(x)-P_1(x)|<1, \quad |f(x)-P_2(x)|<1, \quad \text{and} \quad P_1(x)-P_2(x)\neq\text{const.}$$

in the interval $0\leqq|x|\leqq a$ $(a\geqq 2)$, then we get a polynomial

$$P_1(x)-P_2(x)=C_0x^n+C_1x^{n-1}+\cdots+C_n \qquad (C_0\neq 0)$$

with integral coefficients such that

$$|C_0x^n+C_1x^{n-1}+\cdots+C_n|<2 \quad \text{for} \quad 0\leqq|x|\leqq a;$$

and this contradicts the known theorem of Tschebyscheff [1] that there exists at least one point x in the interval $(-a,\ a)$ for which

$$|C_0x^n+C_1x^{n-1}+\cdots+C_n|\geqq\frac{C_0}{2^{n-1}}a^n\geqq 2C_0.$$

I can not yet find out the upper limit of the intervals to which the theorem can be extended.

(1) Oeuvres, t. 1, pp. 273—378.