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To I. J. Schoenberg and His Mathematics

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To write this evaluation of I. J. Schoenberg's brilliant and variegated contributions to mathematics on the occasion of his three-score and tenth birthday is undoubtedly premature. His distinguished career in mathematical research and scholarship proceeds today with unabated vigor. His devotion, enthusiasm and energy on behalf of mathematics are virtually unbounding. During the past three years alone he made lecture tours and participated in conferences extending over such far reaches as Bangladesh, India, Israel, almost all countries in Europe, Canada, and large parts of the U.S. His lucidity and charm in exposition are well-known.

Born 1903 in Galatz, Rumania, Schoenberg received the M.A. from the University of Jassy, Rumania, in 1922; studied in Göttingen and Berlin from 1922 to 1925; and received the Ph.D. from Jassy in 1926. He came to the United States in 1930; served in various post-doctoral capacities at Chicago and Harvard; was a member of the Institute for Advanced Studies from 1933 to 1935; went on to Swarthmore and Colby Colleges, and in 1941 to the University of Pennsylvania where he remained until 1966 except for periods as a mathematician at Aberdeen Proving Ground during the war in 1943–1945, and various leaves of absence at other institutions. In 1966 he joined the University of Wisconsin as a Professor in the Mathematics Research Center and the Mathematics Department.

Schoenberg's family bears many mathematical connections. His father, although a medical doctor, exhibited a flair for mathematical pastimes. His first, wife (deceased) was the daughter of the eminent mathematician E. Landau. I. J. Schoenberg is a true example of the "established genetic principle" that mathematical talent is an inheritable trait usually transmitted from father to son-in-law. (As a dutiful son-in-law he wrote a number of papers resolving certain problems initiated by Landau; see below.) His brother-in-law was the distinguished H. Rademacher. His present wife, although a concert pianist by training, furnishes charm and wit as a mathematical hostess at the many mathematical conferences to which she accompanies her husband.

Schoenberg is a man of broad culture, fluent in several languages, addicted to art, music and world literature, sensitive, gracious and giving in all ways. Iso (as he is affectionately known to his friends) frequently builds physical models related to his mathematical enquiries. See, for example, his studies on the Kakeya problem [70, 84].[1] The working desk at his home where he engages in research is actually a draftsman's bench complete with a T-square, etc., and a tall stool. Mobiles, artistic works, models of ruled surfaces, icosahedrons and other objects are strewn throughout the room. English, French and German novels, numerous paintings and artifacts are scattered on all the nearby easy chairs. He buys and collects books of all vintages with passion. Historical mathematical discourses especially fascinate him and his articles frequently reflect this interest.

I. J. Schoenberg has made outstanding mathematical contributions in at least three areas of analysis. His early work (already in [4, 15, 17, 19]) pioneered and elaborated the basic concepts of total positivity and variation diminishing transformations. Total positivity is a concept of considerable power that plays an important role in various domains of mathematics, statistics and mechanics. In mathematics, totally positive functions figure prominently (though sometimes indirectly) in problems involving convexity, moment spaces, eigenvalues of integral operators, and the oscillation properties of solutions of linear differential equations, as well as in approximation theory and other areas of (real) analysis. In statistics, the theory of total positivity is fundamental to the understanding of statistical decision procedures, and especially in discerning uniformly most powerful tests for hypotheses involving a finite set of real parameters. Total positivity is also of great importance in ascertaining optimal policies for inventory and production processes, in evaluating the reliability of coherent systems, in the analysis of diffusion-type-stochastic processes, and in the study of vibrating coupled mechanical systems.

The highly important totally positive kernels of the form $K(x, y) = f(x-y)$ where x, y traverse the real line (or the integers), Schoenberg called Pólya frequency functions (sequences). These feature in a wide range of applications to analysis, e.g., they are most pertinent in characterizing those convolution transformations that can be inverted by sequences of polynomial differential operators. For the example of the integral transformation induced by the kernel

$$(1/2\pi t)^{1/2}) \exp[-(x-y)^2/2t] = K(x, y), \qquad t > 0 \text{ fixed}$$

the variation diminishing property is relevant in ascertaining the number of hot and cold pieces of a bar as heat diffuses in the usual pattern. In a

[1] Numbers in square brackets refer to the list of publications following this article.

remarkable series of papers [39, 41, 43, 47, 48, 63], Schoenberg set the basis of the theory of Pólya frequency functions, and established the fundamental representation theorem. Earlier works of Pólya, Laguerre and Schur served as tools for these developments; their concern was the approximation of functions by polynomials with only real zeros. He did collaborate once with Pólya in 1958 to work out some intriguing results pertaining to cyclic Pólya frequency functions (i.e., those defined on the circle) and bearing on certain conjectures concerned with Hadamard products of analytic functions. The characterization of Pólya frequency sequences involves several deep facts from the theory of meromorphic functions [41]. These results led to elegant refinements of the Descartes rule of sign and uncovered a series of remarkable properties of polynomials admitting only negative real zeros [52, 53].

Another line of mathematical activity which occupied Schoenberg was the area of finite distance geometry where he extended and refined ideas of Menger and Blumenthal. His classical work (jointly with von Neumann) [28] concerning screw curves in Hilbert space plays an enticing and significant role in describing the metric geometry of infinite dimensional Euclidean spaces. More importantly, Schoenberg was among the vanguard to offer fundamental and far reaching extensions of the concept of positive definite and completely monotone functions defined on general metric spaces; [23], [25], [26]. For example, the complete characterizations and the representation formula for positive definite functions on spheres, first uncovered by Schoenberg [29], served as the point of departure for much of the further generalizations of positive definiteness to manifolds and symmetric spaces so nicely promulgated by Krein, Yaglom, Gelfand, Gangolli, Ito, and others. These notions found applications and continue to be of basic relevance in the theory of stationary stochastic processes, moment theory and in the study of certain classes of operators in Hilbert spaces.

Beyond the work previously mentioned, Schoenberg is noted worldwide for his realization of the importance of spline functions for general mathematical analysis and in approximation theory, their key relevance in numerical procedures for solving differential equations with initial and/or boundary conditions, and their role in the solution of a whole host of variational problems. The fundamental papers by Schoenberg [31, 32] form a monument in the history of the subject as well as its inauguration.

To illustrate some of its wide scope we cite three specific accomplishments due directly to Schoenberg. He established in [83, 87] the intrinsic connection between monosplines and best quadrature formulas and determined their complete characterization. Recently, in collaboration with Cavaretta [100], he settled an outstanding problem asking to determine sharp bounds on the ν-th derivatives of a function f defined on $(0, \infty)$, i.e., sharp estimates of

$\|f^{(\nu)}\|_\infty$, $1 \leqslant \nu \leqslant n-1$, in terms of $\|f^{(n)}\|_\infty$ and $\|f\|_\infty$. The solution involved a new class of splines: the one-sided Euler splines. The mathematicians who have worked on this type of problems include Landau, Kolmogorov and others. Kolmogorov solved the corresponding problem for the two-sided infinite interval $(-\infty, \infty)$ using the classical Euler splines. The finite interval case remains open.

The theory of cardinal splines ([98], [110], [114]) and their wide ramifications in approximation theory is his full baby. A monograph on this topic authored by Schoenberg is forthcoming.

The subject of spline theory and its applications has mushroomed, yielding results in many domains, e.g., generalized interpolation theory, Sobolev spaces, numerical methods in differential equations, statistical estimation and regression analysis procedures, optimal design, control theory, and so forth. Extension of the concept of spline to several variables remains a challenging area.

A recent bibliographic compilation reveals that over the last 15 years close to 1000 articles have been published related to spline theory. In recent conferences on approximation theory splines constitute a major topic. With all this activity, it is universally accepted that I. J. Schoenberg is "Mr. Spline," as he has been dubbed on many occasions. But apart from fathering the subject, he enthusiastically and persistently contributes to it, profoundly and substantively.

Schoenberg's mathematics is replete with versatility, originality, and richness in ideas and technique. His inspiration, knowledge and insights continue to abet and inspire numerous colleagues and students. I and all of his many friends, in commemoration of his 70th birthday, offer our affection, esteem and best wishes for everlasting good health, more discovery of mathematical beauty and continued joy of creativity.

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