KOLMOGOROV ENTROPY OF THE MIXED DERIVATIVE SPACES

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1. The Mixed Derivative Spaces

Eduard Belinskiĭ's forty-nine papers touch on different aspects of the field of Approximation Theory, and among these papers several directly address questions concerning functions with bounded mixed derivatives, see for instance [2–9]. This is a subject which originated in the work of N. M. Korobov in the 1950's, and attracted a substantial interest over a period of decades. The culmination of much that was learned is contained in the reference [15]. The subject has points of contact with several different areas, and some new results, which this brief note will only touch upon.

For integers $d \geq 2$, let $W_d^{1,2}$ be the functions $F : [0,1] \to \mathbb{R}$ with

$$\|F\|^2_{W_d^{1,2}} := |F(0)|^2 + \int_{[0,1]^d} |\partial_{x_1} \cdots \partial_{x_d} F(x_1, \dots, x_d)|^2 \, dx_1 \cdots dx_d < \infty \, .$$

Thus, these functions have one mixed derivative in L^2. In this setting, there is a Sobolev embedding inequality which shows that $W_d^{1,2} \subset C([0,1]^d)$. Indeed, the unit ball $B_d^{1,2}$ of $W_d^{1,2}$ is a compact subset of $C([0,1]^d)$.

One can then study the Kolmogorov entropy of these spaces with respect to different L^q-metrics. Namely, for some $1 \leq q \leq \infty$, and $0 < \varepsilon < 1$, one takes $N(d,q,\epsilon)$ to be the least number N so that there are functions f_n, $1 \leq n \leq N$ so that

$$B_d^{1,2} \subset \bigcup_{n=1}^{N} f_n + \{g \,:\, \|g\|_q \leq 1\} .$$

These numbers, of a quite classical nature if $d = 1$, are more subtle in dimensions $d \geq 2$. Consider the exponentials

$$e^{2\pi i n \cdot x} = \exp\left(2\pi i \sum_{j=1}^{d} n_j x_j\right) .$$

These are of course eigenfunctions of the mixed derivative, with eigenvalue $(2\pi i)^d \prod_{j=1}^d n_j$. Accordingly, one must group the exponentials together according to the 'hyperbolic' quantity $\prod_{j=1}^d n_j$. This variation has profound implications for the subject.

A central result here concerns the Kolmogorov entropy numbers, when measured with respect to an L^2 norm. (Both in the definition in the mixed derivative space, and the statements of the Theorems, we do not strive for the greatest generality.)

Theorem 1.1. *For $d \geq 2$, we have*

$$\log N(d, 2, \varepsilon) \simeq \varepsilon^{-1}(\log 1/\varepsilon)^{d-1}, \qquad \varepsilon \downarrow 0 .$$

Note that the dimension enters into the power of the logarithm. At the endpoint of $q = \infty$, there is expected to be a jump in the behavior.

Conjecture 1.2. *For $d \geq 2$ one has the estimate*

$$\log N(d, \infty, \varepsilon) \simeq \varepsilon^{-1}(\log 1/\varepsilon)^{d-1/2}, \qquad \varepsilon \downarrow 0 . \tag{1.3}$$

The upper bound in (1.3) is known, and the part that remains is to establish the lower bound, that is, to construct large collections of functions in the mixed derivative space, of small norm, which are provably far apart in the L^∞-norm.

This is resolved in dimension $d = 2$ after an essential contribution of Talagrand. See [3, 13, 14]. On the other hand, this conjecture is open in dimensions $d = 3$. The difficulty of the conjecture arises from the behavior of sums of hyperbolic exponentials

$$\sum_{n \in H_j} a_n \cdot e^{2i\, n \cdot x}$$

where H_j denote the set of integers $n \in \mathbb{N}^d$ with $2^j \leq \prod_{j=1}^d n_j < 2^{j+1}$. Here, the coefficients a_n can be thought of a bounded sequence, say either $-1, 0, 1$. The L^∞-norm of such a sum is conjecturally, larger than what is predicted by the average case analysis, and the L^∞-norm is achieved on a set that is exponentially small in n—hence invisible to the average case analysis. There is a change in the nature of this gain over the average case analysis as one passes from two to three dimensions, making the case of three dimensions much harder.

The underlying difficulty in this conjecture is closely related to the difficulties in related subjects. One of these is so-called Small Ball Probabilities for certain Gaussian processes, a connection which arose from Talagrand's result, (See [12]) and one that Belinskiĭ made contributions too ([3]). Another is the the Theory of Irregularity of Distributions ([1]), which subject, with it's connections to the Mixed Derivative space are described in [16]. Indeed, the range of connections also hints of a range of methods that have been brought to bear on this question.

Recently, there has been some progress on this conjecture. For this theorem, see [10], also [11].

Theorem 1.4 (Bilyk, L., Vagharshakyan)**.** *In dimensions $d \geq 3$ there is an $\eta = \eta(d) > 0$ so that we have the estimate*

$$\log N(d, \infty, \varepsilon) \gtrsim \varepsilon^{-1}(\log 1/\varepsilon)^{d-1+\eta}, \qquad \varepsilon \downarrow 0 .$$

This proof has an *ad hoc* nature that is not satisfactory. Nevertheless, one can hope that some of the techniques used will inspire those that are required to solve the full conjecture.

2. A Personal Remembrance

I met Eduard Belinskiĭ only once, on a conference in 1996, leaving a hot Atlanta summer, for a brief stay in the beautiful light of a South African winter. We both attended the first joint meeting between the American Mathematical Society and some societies of African mathematicians, in

Pretoria. This was the time that Eduard was working in Harare, Zimbabwe. We met at the analysis special session, and had several stimulating conversations, as of course our interests strongly overlapped. Unfortunately, he had to return to Harare shortly after the meeting, and while we corresponded in the intervening years, we never met again.

He passed away before my interest in the mixed derivative questions truly developed. We would have had a great time discussing this problem, and the insights he would have had into these new developments are sadly lost to us.

References

[1] József Beck and William W. L. Chen, *Irregularities of distribution*, Cambridge Tracts in Mathematics, vol. 89, Cambridge University Press, Cambridge, 1987.MR903025 (88m:11061) ↑2

[2] È. S. Belinskiĭ, *Estimates for Kolmogorov widths of classes of functions with conditions for the mixed difference in the uniform metric*, Mat. Zametki **50** (1991), no. 5, 147–149 (Russian); English transl., Math. Notes **50** (1991), no. 5-6, 1198–1200 (1992). MR **1155567 (92m:**41060) ↑1

[3] Eduard Belinskiĭ and Werner Linde, *Compactness properties of certain integral operators related to fractional integration*, Math. Z. **252** (2006), no. 3, 669–686. MR **2207763 (2007e:**47073) ↑1, 2

[4] ______, *Small ball probabilities of fractional Brownian sheets via fractional integration operators*, J. Theoret. Probab. **15** (2002), no. 3, 589–612. MR **1922439 (2004d:**60092) ↑1

[5] È. S. Belinskiĭ, *Approximation of functions of several variables by trigonometric polynomials with given number of harmonics, and estimates of ϵ-entropy*, Anal. Math. **15** (1989), no. 2, 67–74 (English, with Russian summary). MR **1030345 (90m:**42001) ↑1

[6] ______, *Approximation by a "floating" system of exponentials on classes of periodic functions with a bounded mixed derivative*, Studies in the theory of functions of several real variables (Russian), Matematika, Yaroslav. Gos. Univ., Yaroslavl′, 1988, pp. 16–33 (Russian). MR 1166391 ↑1

[7] È. S. Belinskiĭ and I. R. Liflyand, *The behavior of Lebesgue constants of hyperbolic partial sums*, Mat. Zametki **43** (1988), no. 2, 192–196, 300 (Russian); English transl., Math. Notes **43** (1988), no. 1-2, 107–109. MR **939519 (89g:**42024) ↑1

[8] È. S. Belinskiĭ, *Trigonometric diameters of some classes of smooth functions of several variables*, Theory of functions and approximations, Part 2 (Russian) (Saratov, 1984), Saratov. Gos. Univ., Saratov, 1986, pp. 39–42 (Russian). MR 990930 ↑1

[9] E. S. Bēlīns′kiĭ, *Some properties of hyperbolic partial sums*, Theory of functions and mappings (Russian), "Naukova Dumka", Kiev, 1979, pp. 28–36, 175 (Russian). MR **568044 (81f:**42016) ↑1

[10] Dmitriy Bilyk, Michael T. Lacey, and Armen Vagharshakyan, *On the small ball inequality in all dimensions*, J. Funct. Anal. **254** (2008), no. 9, 2470–2502. MR 2409170 ↑2

[11] Dmitriy Bilyk and Michael T. Lacey, *On the small ball inequality in three dimensions*, Duke Math. J. **143** (2008), no. 1, 81–115. MR 2414745 ↑2

[12] James Kuelbs and Wenbo V. Li, *Metric entropy and the small ball problem for Gaussian measures*, J. Funct. Anal. **116** (1993), no. 1, 133–157.MR 94j:60078 ↑2

[13] Michel Talagrand, *The small ball problem for the Brownian sheet*, Ann. Probab. **22** (1994), no. 3, 1331–1354.MR 95k:60049 ↑2

[14] V. N. Temlyakov, *An inequality for trigonometric polynomials and its application for estimating the entropy numbers*, J. Complexity **11** (1995), no. 2, 293–307.MR 96c:41052 ↑2

[15] ______, *Approximation of functions with bounded mixed derivative*, Proc. Steklov Inst. Math. (1989), no. 1(178), vi+121. MR **1005898 (90e:**00007) ↑1

[16] ______, *Cubature formulas, discrepancy, and nonlinear approximation*, J. Complexity **19** (2003), no. 3, 352–391. Numerical integration and its complexity (Oberwolfach, 2001). MR **1984119 (2004c:**41037) ↑2

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