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The Work of G. G. Lorentz

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1. HISTORICAL SURVEY

My scientific career at the University of Leningrad (1931–1941) started rather slowly. One of the reasons may have been that I was not a candidate for a degree and for many years I had an enormous teaching load. In 1944 I came to Tübingen as an assistant to E. Kamke who had been dismissed from the University of Tübingen, but was allowed to write books. Despite poor living conditions, the years 1945–1948 in Germany were very successful for me, especially 1946 when I wrote, among others, [17], [20] and [24]. In Tübingen I collaborated with Kamke [13] and Knopp [23]. In the boom after the war I have had some of my best Ph.D. students: W. Jurkat and K. L. Zeller in Tübingen; G. M. Petersen and P. L. Butzer in Toronto. Another successful period is the one which began in 1967 at the University of Texas at Austin, not only because of my collaboration with excellent mathematicians (Berens, Schumaker, Shimogaki, Zeller), but also because of the congenial atmosphere at this University.

2. WORK IN SUMMABILITY THEORY

Paper [1] had been written while I was a student. I had been very proud of it, until it was discovered, after its publication, that the main results were not new (W. Hahn, others). Paper [17] introduced and discussed almost convergence which has since been much applied and generalized. It had as predecessors papers [5] and [8]. It contained some of the results of Agnew about general summability methods (found also by myself during the war but published after the appearance of Agnew's papers), including a first nonequivalence theorem. The methods of the paper were used in [22], [26] and [39] to describe conditions which are not Tauberian. Paper [37] contained a satisfactory general theory of Tauberian theorems for absolute summability. Unfortunately, the theorems have been stated for o-conditions, while proved for O-conditions. Papers [42] and [43] would not have been possible without the genius of K. L. Zeller, with whom I collaborated. The first paper introduced a new functional-analytic method for the treatment of two (or several) summability fields, and the notion of singularity of a pair of matrices. G. M. Petersen has continued this work with success. Paper [43] determined Steinitz sets of regular matrices A. A set $C \subset \mathbb{C}$ is called a Steinitz set of A if there is a sequence s_n such that the A-limits of all A-summable rearrangements of s_n are precisely C. Erdös has determined the Steinitz sets of the Cesàro method C_1. Here, the Steinitz sets of all A were proved to be analytic subsets of $\mathbb{C}$. The direct theorem had a reasonably simple proof, but the inverse one required a monstrous construction.

3. Number Theory

Paper [38] solved a problem suggested by Erdös, and gave a counterpart to the theorem of Mann; see the book of Halberstam and Roth entitled "Sequences." According to Čebyšev, the probability that m, n are relatively prime is $6\pi^{-2}$. It is more difficult to obtain this result if the pair m, n is restricted in some way. This result was proved in [44] when m is the integral part, $[g(n)]$, of a function $g(x)$ which increases more slowly than $x/\log \log x$, but faster than $\log \log \log \log x$. The proof is heavily number theoretic.

4. Approximation Theory

(a) *Bernstein polynomials.* The main theorem in [4], that the derivative of $B_n(f)(x)$ converges to the corresponding derivative of $f(x)$, was not quite easy to prove at that time. Later I gave a simplification in a section devoted to Bernstein polynomials, of a book by Ju. S. Besicovitch. The invention of S. Bernstein continued to inspire me in my later work, directly, [58], [73] and [86] or indirectly, [51], [53], [75] and [77].

(b) *Papers related to the work on metric entropy and widths by Kolmogorov and Vituškin.* The work [52] includes an exposition of the results of the Russian school. It was awarded a prize by the Mathematical Association of America. In [49], widths of many compact sets of functions were determined. These results have often been used, for example, by D. J. Newman. A major paper was [61]: here a new method for computing metric entropy (or ϵ-entropy) was given. Roughly, the entropy of a compact set can be found each time its approximation properties are known. The geometric method used was different from the special methods of Kolmogorov (for classes of functions with $f^{(r)} \in \mathrm{Lip}\,\alpha$) and of Vituškin (for analytic functions), but contained their results as special cases. It also gave many new results, and was not restricted to the uniform metric, as were the earlier methods. Another feature of this paper was the presentation of new proofs of Vituškin's lower estimates of the degree of nonlinear approximation, replacing the extremely difficult methods which were based on multidimensional variations (see Vituškin's book "Theory of Transmission and Processing of Information," Pergamon Press, 1961). Most, but not all, of his results were obtained, together with some new ones. Also in [84], methods of entropy theory are used.

(c) *Positive Coefficients.* Papers [51], [53] and [64] are devoted to polynomials with positive coefficients; the first two raise questions that have not yet been completely answered. The result obtained in [64] bears resemblance to a theorem of Szegö, but the proof is quite different.

(d) *Saturation.* Paper [58] contains the saturation theorem for Bernstein polynomials; this is the complete version of the theorem, a first result being due to De Leeuw. Afterwards, several students of Sunouchi used my method. Several proofs are now available, see for example [75]. Paper [73] is characterized by an ingenious use of intermediate spaces (by H. Berens).

(e) *Monotone Approximation.* Monotone approximation is the approximation of functions by polynomials P_n of degree n satisfying $P_n^{(k)}(x) \geq 0$ for some fixed k. It is interesting that the proof of uniqueness of monotone polynomials of best approximation, [70], depends upon the regularity of certain Birkhoff interpolation problems. This connection has been exploited by some later writers (for example, my son R. A. Lorentz). Determination of the degree of approximation (see [65, 67, 74]) is in many cases possible by means of a smoothed out version of Jackson's integral.

(f) *Birkhoff Interpolation Problems.* Zeller and I became interested in these problems through studying monotone approximation. The work began with a refutation of the Atkinson–Sharma conjecture on the regularity of incidence matrices E, [71]. Later work proved the validity of the conjecture in some restricted situations; for example, when one row of the matrix contains exactly one odd supported sequence [76], or when E is almost simple [82]. In the first case, a method of "Rolle-independent" knots has been used; in the second, a development of the method of coalescence of rows, due to Karlin and Karon. The proofs of the two theorems proved to be unexpectedly difficult. A first full proof of the strong singularity in the first theorem (which seems to require the method of Rolle-independent knots) will appear in [90]. Paper [81] gives a justification of the coalescence method for arbitrary systems of functions, while [83] generalizes a theorem of G. D. Birkhoff to a result about the number of zeros of an arbitrary spline function. The report [90] on the Birkhoff interpolation problem is almost completed, and hopefully will appear soon.

(g) *Korovkin Sets and Shadows.* This work has mainly been done jointly with H. Berens. In [79], a very satisfactory theory has been developed for the spaces L^1, based on a use of Banach lattices. In [78], the problem is approached from a different angle, using approximations which first appeared in [17], and later used by H. Bauer in Choquet theory. More general spaces can be treated this way, but with less general test sets; see also {3}.

5. Functional Analysis

Some of the questions discussed in Section 4 belong partly or predominantly here; for example those in [61], [78], [79]. Papers [19], [24] and [27] introduced the function spaces $\Lambda(\alpha; r)$, $M(\alpha; r)$, much used in present day analysis. Spaces $L^{p,q}$, almost identical to $\Lambda(\alpha; r)$ spaces have been called Lorentz spaces by Calderón. They are particular instances of Köthe spaces and are closely connected with the theory of intermediate spaces. Also the classical L^p spaces are intermediate spaces between some $\Lambda(\alpha; r)$ spaces. This work has been continued in [34], [45], [48] and [50]. In 1955 I prepared a paper dealing with the general theory of Banach function spaces. It contained material in common with the Ph.D. dissertation of W. A. J. Luxemburg, written at the same time. For this reason I withdrew my paper, but Luxemburg and Zaanen have quoted it in a most generous fashion. Papers [63] and [72] were devoted to interpolation theorems for linear operators. The method developed there was based on certain pseudo-inequalities for functions, not unlike the relation of Hardy, Littlewood and Pólya. This method is difficult to apply, but if it works, it gives good results, sometimes even the exact constants.

6. Real Functions

Papers [32] and [40] give inequalities of a Hardy, Littlewood and Pólya type. The original inequalities were bilinear, but the new ones contain an arbitrary function of several variables. Paper [20] described conditions which the measures of the cross-sections of an arbitrary plane measurable set must satisfy. It has been discovered recently that the Gale–Ryser theorem in the books on Combinatorial Analysis (a theorem about matrices of 0's and 1's), found much later, is a special case of my result. Papers [13] and [21] concern Dirichlet's problem; paper [12] deals with classes of functions and Fourier coefficients. Its original version contained some results of Hardy and Rogosinski on lacunary Fourier series obtained during the war and published before this paper appeared.

7. Books

This book on Bernstein polynomials [86] contains several new ideas. The most important ones are probably the convergence results, $B_n f \to f$, in complex domains, which go beyond the original papers of S. Bernstein. It was hoped that this book would provide a pattern valid for similar theorems for other operators. It contains a solution of the Hausdorff moment problem for arbitrary Banach function spaces, based on an interpolation theorem. The proof of the latter could have also been applied to give the Calderón–Mitjagin theorem found later (in 1966). A new form of Okada's theorem, a theory of Banach function spaces, in particular $\Lambda(\alpha; r)$ and $M(\alpha; r)$, and a new proof of an inequality of Hardy, Littlewood and Pólya also appear in this book "Approximation of Functions" [87]. My idea in this short book was to represent all important branches of Approximation Theory (except for approximation in the complex domain and splines) by means of the significant and often even best possible theorems. Some of the material (entropy, widths, representation of functions by superpositions) had not appeared in book form before. Unfortunately, the book is not commercially available as of the time of this writing. Korovkin Sets ("Sets of Convergence") [88]. These lecture notes were distributed during the regional conference in Riverside in 1972. Beside the results from [78] and [79], they contain a new exposition of Šaškin's theory of Korovkin sets for the space of continuous functions. "Approximation Theory" [89]. It was a great labor to obtain funds for the International Symposium on Approximation Theory in Austin, Texas, January 1973, to ensure that it ran smoothly, and to prepare the manuscripts for publication. An exceptionaly large share of the latter work was taken over by H. Berens.

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