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JÓZEF MARCINKIEWICZ (1910–1940) – ON THE CENTENARY OF HIS BIRTH

LECH MALIGRANDA

Department of Engineering Sciences and Mathematics
Luleå University of Technology, SE-971 87 Luleå, Sweden
E-mail: lech.maligranda@ltu.se

Abstract. Józef Marcinkiewicz's (1910–1940) name is not known by many people, except maybe a small group of mathematicians, although his influence on the analysis and probability theory of the twentieth century was enormous. This survey of his life and work is in honour of the 100th anniversary of his birth and 70th anniversary of his death. The discussion is divided into two periods of Marcinkiewicz's life. First, 1910–1933, that is, from his birth to his graduation from the University of Stefan Batory in Vilnius, and for the period 1933–1940, when he achieved scientific titles, was working at the university, did his army services and was staying abroad. Part 3 contains a list of different activities to celebrate the memory of Marcinkiewicz. In part 4, scientific achievements in mathematics, including the results associated with his name, are discussed. Marcinkiewicz worked in functional analysis, probability, theory of real and complex functions, trigonometric series, Fourier series, orthogonal series and approximation theory. He wrote 55 scientific papers in six years (1933–1939). Marcinkiewicz's name in mathematics is connected with the Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, the Marcinkiewicz integral and function, Marcinkiewicz–Zygmund inequalities, the Marcinkiewicz–Zygmund strong law of large numbers, the Marcinkiewicz multiplier theorem, the Marcinkiewicz–Salem conjecture, the Marcinkiewicz theorem on the characteristic function and the Marcinkiewicz theorem on the Perron integral. Books and papers containing Marcinkiewicz's mathematical results are cited in part 4 just after the discussion of his mathematical achievements. The work ends with a full list of Marcinkiewicz's scientific papers and a list of articles devoted to him.

2010 *Mathematics Subject Classification*: Primary 01A70, 46E30, 46B70, 28A15, 60E15, 46B09, 60E10, 60F15, 42B15, 26D05; Secondary: 41A10, 26D15, 01A60.

Key words and phrases: Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, Marcinkiewicz integral, independent random variables, inequalities in probability, characteristic functions, differentiation theory, maximal functions, multipliers, Marcinkiewicz sets, convergence of Riemann sums, vector-valued inequalities, Fourier and orthogonal series, Lagrange interpolation, inequalities for trigonometric functions and polynomials, rearrangements of series, universal primitive functions, Perron integral.

The paper is in final form and no version of it will be published elsewhere.

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1. Life of Marcinkiewicz from the birth to university (1910–1933). Józef Marcinkiewicz was born on 12 April 1910 (30 March 1910 in old style=Julian calendar) in the small village Cimoszka near Białystok (Poland). His parents were Klemens Marcinkiewicz (1866–1941) and Aleksandra Marcinkiewicz née Chodakiewicz (1878–1941). Józef was the fourth of five children.

The children of Klemens and Aleksandra Marcinkiewicz were: Stanisława Marcinkiewicz–Lewicka (1903–1988), Mieczysław Marcinkiewicz (1904–1976), Edward Marcinkiewicz (1908–1985), Józef Marcinkiewicz (1910–1940), Kazimierz Marcinkiewicz (1913–1946).

Photo 1. Józef Marcinkiewicz and his signature

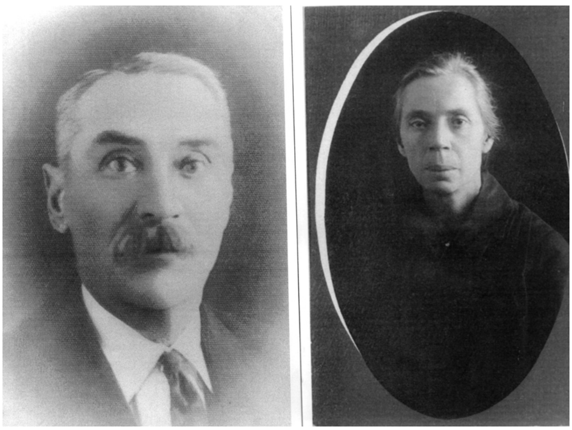
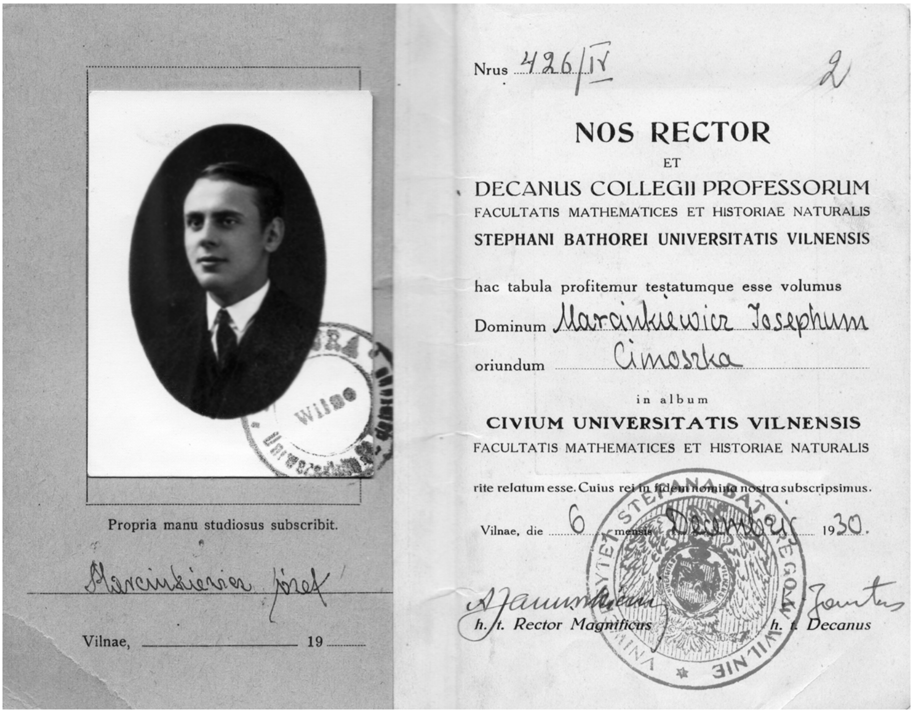


Photo 2. Parents of Józef Marcinkiewicz

Marcinkiewicz grew up with some health problems, in particular he had lung trouble, but this did not prevent him taking an active part in sports. Swimming and skiing were two sports at which he became particularly proficient. Because of his poor health, Marcinkiewicz first took private lessons at home and then he finished elementary school in Janów.

After that Marcinkiewicz went to the District Gymnasium in Sokółka (after 4^{th} class of elementary school and examination). In the period 1924–1930 he studied at the King Zygmunt August State Gymnasium in Białystok. He obtained his secondary-school certificate (matura certificate) on 22 June 1930 (number 220/322).



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ET

DECANUS COLLEGII PROFESSORUM

FACULTATIS MATHEMATICES ET HISTORIAE NATURALIS

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FACULTATIS MATHEMATICES ET HISTORIAE NATURALIS

rite relatum esse. Cuius rei in fidem nomina nostra subscripsimus.

Vilnae, die 6 mensis Decembris 1930.

h. t. Rector Magnificus h. t. Decanus

Propria manu studiosus subscribit.

Marcinkiewicz Józef

Vilnae, 19

Photo 3. Photo and the first page of Marcinkiewicz's gradebook ("indeks")

In 1930 Marcinkiewicz became student at the Department of Mathematics and Natural Science, Stefan Batory University (USB) in Wilno (then in Poland, now Vilnius in Lithuania). From the first year at the University, Marcinkiewicz demonstrated knowledge of the subject and exceptional mathematical talent. He attracted the attention of the following three professors from the Department: Stefan Kempisty, Juliusz Rudnicki and Antoni Zygmund.[1]

[1]Stefan Jan Kempisty (born 23 July 1892 in Zamość – died 5 August 1940 in prison in Wilno), Juliusz Rudnicki (born 30 March 1881 in village Siekierzyńce near Kamieniec Podolski – died 26 February 1948 in Toruń), Antoni Zygmund (born 25 December 1900 in Warsaw – died 30 May 1992 in Chicago).

In his second year of studies he participated, in the academic year 1931/32, at Zygmund's course on *orthogonal series* preceded by an *introduction to the theory of Lebesgue integration.* This course was too difficult for the average second year student and Marcinkiewicz asked Zygmund for permission to take this course. That was the beginning of their fruitful mathematical collaboration.

As well as Marcinkiewicz, there were a few other young mathematicians, for example, Konstanty Sokół-Sokołowski, later he obtained PhD in mathematics and senior assistant at the Department of Mathematics, who became like Marcinkiewicz one of the victims of the war (and like Marcinkiewicz he was killed in Kharkov in 1940).[2]

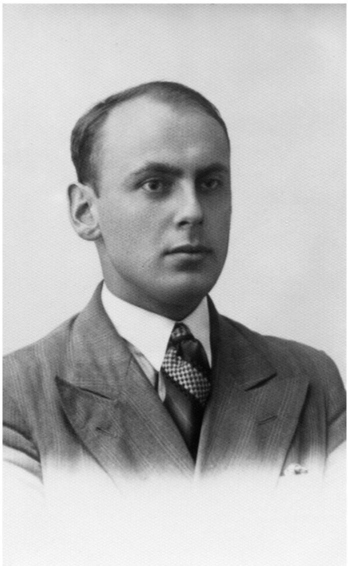


Photo 4. Józef Marcinkiewicz

Zygmund wrote ([JMCP], pp. 2–3):

> *When I think of Marcinkiewicz I see in my imagination a tall and handsome boy, lively, sensitive, warm and ambitious, with a great sense of duty and honor. He did not shun amusement, and in particular was quite fond of dancing and the game of bridge. His health was not particularly good; he had weak lungs and had to be careful of himself. He was interested in sports (possibly because of his health) and was a good swimmer and skier. He also had intellectual interests outside Mathematics, knew a lot of modern Physics and certain branches of Celestial Mechanics. He said to me once that before entering the university he had hesitated about whether to choose mathematics or Polish literature.*

[2]Konstanty Sokół-Sokołowski (born 9 September 1906 in Tarnobrzeg – killed in April or May 1940 in Kharkov). He has written his PhD thesis under supervision of Zygmund in 1939 *On trigonometric series conjugate to Fourier series of two variables.*

We must remember that Antoni Zygmund was not only a master for Marcinkiewicz, but also an advocate for his achievements. In 1940 Zygmund emigrated to the United States, and from 1947 he worked in Chicago, where he created the famous school of mathematics. His students were among others Alberto Calderón (1920–1998), Leonard D. Berkovitz (1924–2009), Paul J. Cohen (1934–2007) – awarded the Fields Medal in 1966, Mischa Cotlar (1912–2007), Eugene Fabes (1937–1997), Nathan Fine (1916–1994), Benjamin Muckenhoupt (1933), Victor L. Shapiro (1924), Elias Stein (1931), Daniel Waterman (1927), Guido Weiss (1928), Mary Weiss (1930–1966), Richard Wheeden (1940) and Izaak Wirszup (1915–2008). It is thanks to Zygmund, who survived the war and became an important mathematician in the world, the name of Marcinkiewicz also became widely known among mathematicians.

Marcinkiewicz was interested in literature, music, painting, poetry, and he also wrote poetry himself. He liked almost all areas of life and also to talk about different topics. While studying he learnt English, French and Italian. Mathematics, however, he always put first. He took an active part in the student life participating in various events organized by the Mathematical-Physical Circle; in the academic year 1932/33 he was president of the Board. His closer friends and colleagues were: Stanisław Kolankowski, Wanda Onoszko, Danuta Grzesikowska (-Sadowska) and Leon Jeśmanowicz.

Marcinkiewicz graduated in 1933, after only three years of study and on 20 June 1933 he obtained a Master of Science degree (in mathematics) at USB. The title of his master thesis was *Convergence of the Fourier–Lebesgue series* and the supervisor was Professor Antoni Zygmund.

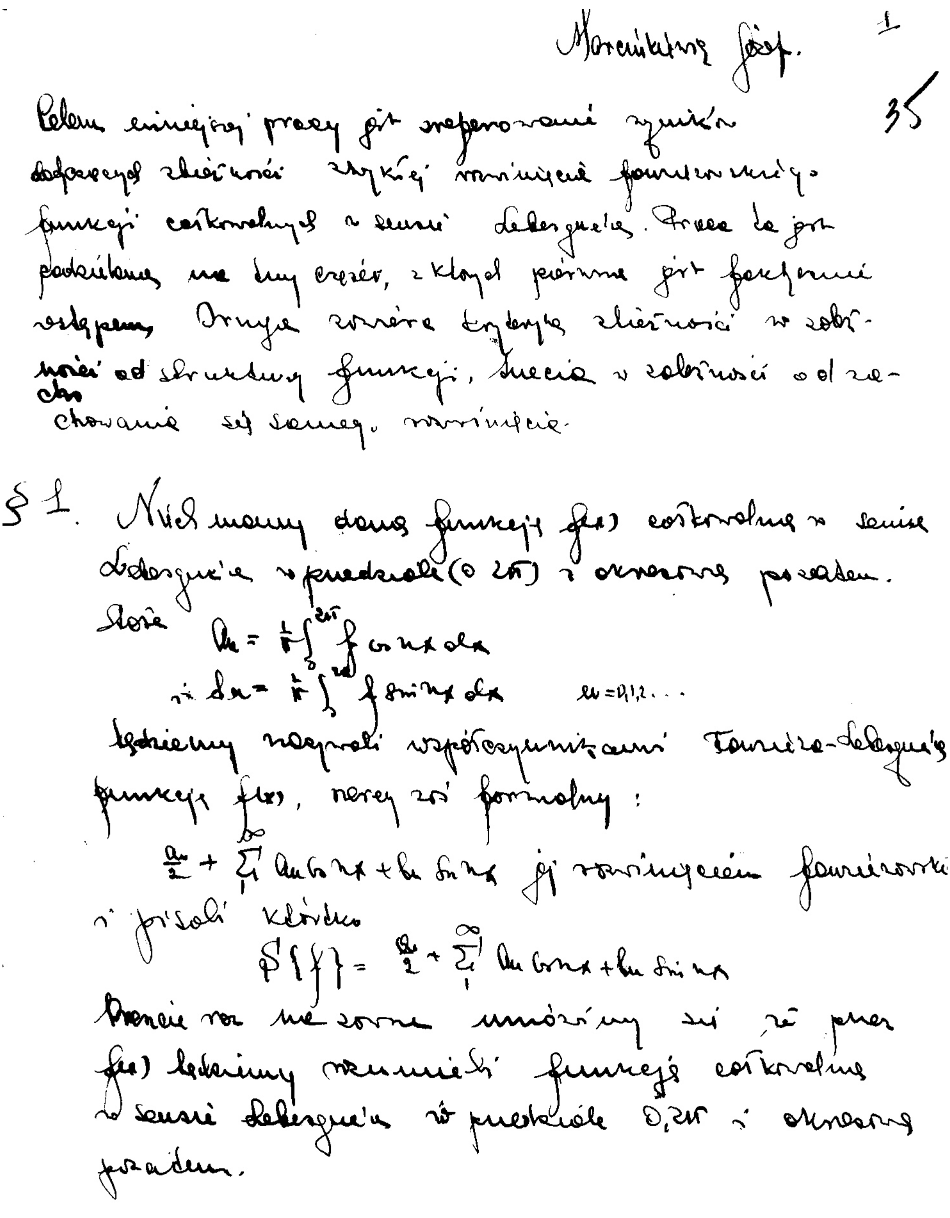
His M.Sc. thesis consisted of his first original results in mathematics and contained, among other things, the proof of the new and interesting theorem that there exists a continuous periodic function whose trigonometric interpolating polynomials, corresponding to equidistant nodal points, diverge almost everywhere. These results, in a somewhat extended form, were presented two years later as his PhD thesis.

2. Scientific career, work, military service and tragic end. In the periods September 1933 – August 1934 and September 1934 – August 1935 he was assistant to the Zygmund chair of mathematics at USB.

In the interim he did one year military service in 5$^{\text{th}}$ Infantry Regiment of Legions in Wilno. He finished the military course with an excellent score. On 17 September 1934 he was transferred to the reserves. Marcinkiewicz took his soldiering duties seriously, but not without a sense of humour as far as the disadvantages of military service were concerned. He received the following evaluation:

> *Outstanding individuality. Very energetic and full of initiative.* (...) *Deep and bright mind.* (...) *Memory and logical thinking very good* (...) *Characterized by a good planning and persistence in work. Overall evaluation*: *outstanding.*

In September 1934 he returned to USB and on 25 June 1935 he defended his PhD thesis entitled *Interpolation polynomials of absolutely continuous functions* at the Stefan Batory University in Wilno, under supervision of Antoni Zygmund. His PhD thesis was a booklet



Marcinkiewicz Józef. 1

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Celem niniejszej pracy jest zreferowanie wyników dotyczących zbieżności znykłego rozwinięcia fourierowskiego funkcji całkowalnych w sensie Lebesgue'a. Praca ta jest podzielona na trzy części, z których pierwsza jest jakby wstępem. Druga zawiera kryteria zbieżności w zależności od struktury funkcji, trzecia w zależności od zachowania się samego rozwinięcia.

§ 1. Niech mamy daną funkcję f(x) całkowalną w sensie Lebesgue'a w przedziale (0 2π) i okresową poza tem.
Kładę
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f \cos nx\, dx$$
$$\text{i } b_n = \frac{1}{\pi}\int_0^{2\pi} f \sin nx\, dx \qquad n = 0, 1, 2, \ldots$$
będziemy nazywali współczynnikami Fourier-Lebesgue'a funkcji f(x), szereg zaś formalny:
$$\frac{a_0}{2} + \sum_1^\infty a_n \cos nx + b_n \sin nx$$ jej rozwinięciem fourierowskim i pisali krótko
$$S\{f\} = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + b_n \sin nx$$
Ponieważ raz na zawsze umówimy się, że przez f(x) będziemy rozumieli funkcję całkowalną w sensie Lebesgue'a w przedziale 0,2π i okresową poza tem.

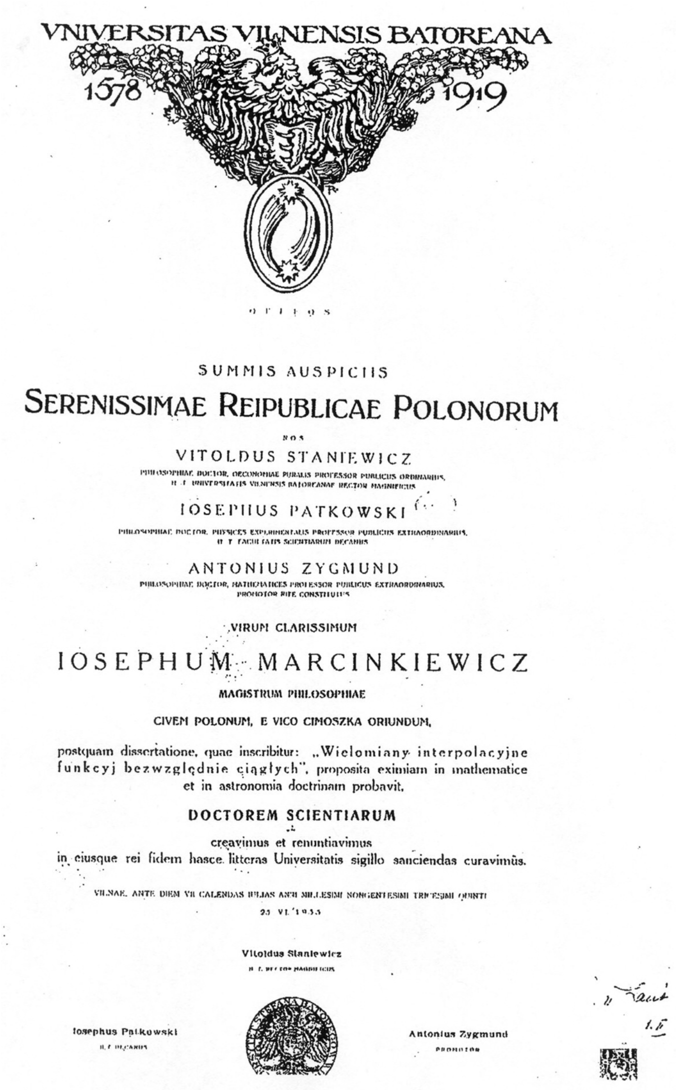
Photo 5. 1933. The first page of hand-written master thesis of Józef Marcinkiewicz

of 41 pages published in Polish in Dissertationes Inaugurales No. 10, USB. It was also published as the paper in [M35c] and its English translation appeared in [JMCP], pp. 45–70. The evaluation of Marcinkiewicz's PhD dissertation made by Zygmund (28 May 1935) contains the following opinion:

> *I think Marcinkiewicz's work is very valuable, showing big mathematical talent and originality of the author. I accept it as a doctoral dissertation.*

Photo 6. XI Congress of Scientific Mathematical-Physical and Astronomical Circles (25–28 May 1933) in Wilno. Congress Hall USB. Sitting in the first row from the left: Kazimierz Jantzen (1885–1940), Józef Stanisław Patkowski (1887–1942) – physicist, Kazimierz Opoczyński (1877–1963) – rector of USB, Władysław Dziewulski (1878–1962) – astronomer, Józef Marcinkiewicz (1910–1940), Ira Anna Koźniewska (1911–1989) – statistician, Wacław Michał Dziewulski (1882–1938) – physicist, Stefan Jan Kempisty (1892–1940), Bogumił Jasinowski (1883–1969) – philosopher, Aleksander Januszkiewicz (1872–1955), Antoni Zygmund (1900–1992), Edward Szpilrajn-Marczewski (1907–1976)

At the PhD examination, taken on 7 June 1935, the questions to Marcinkiewicz were the following: Zygmund: *problem of approximation of functions, interpolation theory (Legendre, Hermite), quadratic approximation, Chebyshev polynomials, Fejér results, convergence criterion in the case when nodes are zeros of Jacobi polynomials, means approximation of* $p \neq 2$; Rudnicki: *entire functions, results of Weierstrass, Poincaré, Borel, Picard, Hadamard, theory of Nevanlinna and Julia*; Kempisty: *Perron integral and its different definitions, similar question for functions of two variables, surface and its measuring, results of Rado and Tonelli*; Dziewulski: *equations of motion in mechanics of celestial bodies, integrals of these equations, the perturbation function, Lagrangian points, motion of the stars and the currents of stars.*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SUMMIS AUSPICIIS
SERENISSIMAE REIPUBLICAE POLONORUM
NOS
VITOLDUS STANIEWICZ
PHILOSOPHIAE DOCTOR, OECONOMIAE RURALIS PROFESSOR PUBLICUS ORDINARIUS,
H. T. UNIVERSITATIS VILNENSIS BATOREANAE RECTOR MAGNIFICUS
IOSEPHUS PATKOWSKI
PHILOSOPHIAE DOCTOR, PHYSICES EXPERIMENTALIS PROFESSOR PUBLICUS EXTRAORDINARIUS,
H. T. FACULTATIS SCIENTIARUM DECANUS
ANTONIUS ZYGMUND
PHILOSOPHIAE DOCTOR, MATHEMATICES PROFESSOR PUBLICUS EXTRAORDINARIUS,
PROMOTOR RITE CONSTITUTUS
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IOSEPHUM MARCINKIEWICZ
MAGISTRUM PHILOSOPHIAE
CIVEM POLONUM, E VICO CIMOSZKA ORIUNDUM,
postquam dissertatione, quae inscribitur: „Wielomiany interpolacyjne funkcyj bezwzględnie ciągłych", proposita eximiam in mathematice et in astronomia doctrinam probavit,
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H. T. DECANUS
Antonius Zygmund
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Photo 7. Diploma of doctor of philosophy

During the years 1934–1938 Marcinkiewicz was taken on six-weeks military exercises (25 June–16 September 1934, 12 August–21 September 1935, 1 July–10 August 1936, 1938 – before travel to France). The spent the academic year 1935/1936 at the Jan Kazimierz University in Lwów. This was a one year Fellowship from the Fund for National Culture and the assistant position at the chair of Stefan Banach in the period 1 December 1935–31 August 1936 with 12 hours of teaching weekly (cf. [DP10], pp. 59–61). Marcinkiewicz visited the *Scottish Café*. He solved problems 83 of Auerbach, 106 of Banach and 131 of Zygmund from the *Scottish Book*. Moreover, he posed his own problem number 124 (cf.

[Mau81], pp. 211–212). In Lwów Marcinkiewicz cooperated with Juliusz Paweł Schauder (1899–1943), who had returned to Lwów a year earlier having spent time in Paris working with Hadamard and Leray.[3]

Photo 8. Józef Marcinkiewicz

Zygmund writes ([JMCP], p. 3):

> *The influence of Schauder was particularly beneficial and would probably have led to important developments had time permitted. For in the field of real variable Marcinkiewicz had exceptionally strong intuition and technique, and the results he obtained in the theory of conjugate functions, had they been extended to functions of several variables might have given (as we see clearly now) a strong push to the theory of partial differential equations. The only visible trace of Schauder's influence is a very interesting paper of Marcinkiewicz on the multipliers of Fourier series, a paper which originated in connection with a problem proposed by Schauder (...)*

While in Lwów Marcinkiewicz also collaborated with Stefan Kaczmarz (1895–1939) and Władysław Orlicz (1903–1990)[4] he became interested in problems of general orthogonal systems and wrote a series of papers on this subject. He published joint paper with Kaczmarz on multipliers of Fourier series and was working in Lwów on general orthogonal series.

Marcinkiewicz was nominated senior assistant to the chair of mathematics at USB for the period 1 September 1936 – 31 August 1937 and on 16 April 1937 Marcinkiewicz filled in an application to commence his habilitation. After one month, on 25 May 1937 Zygmund wrote the following opinion about the papers of Marcinkiewicz:

[3] Juliusz Paweł Schauder (born 21 September 1899 in Lwów – killed in September 1943).

[4] Stefan Kaczmarz (born 20 March 1895 in Sambor – killed in September 1939), Władysław Orlicz (born 24 May 1903 in Okocim – died 9 August 1990 in Poznań).

From the above discussion the work of Dr. Marcinkiewicz shows that it contains a number of interesting and important results. Some of them, due to their final form, will certainly appear in textbooks in mathematics. It should be mentioned that in some of the early papers we can already see strong and subtle arithmetic techniques; things of rare quality. The entire collection is extremely favorable and testifies to the multilateral and original mathematical talent of the author.

Photo 9. Wilno, 4 March 1936. Doctor *honoris causa* for Professor Kazimierz Sławiński[5]. In the foreground (from the left): Juliusz Rudnicki, N.N., Kornel Michejda, Stefan Kempisty, Edward Bekier. At the wall (fourth and fifth from the left): Józef Marcinkiewicz and Antoni Zygmund

Marcinkiewicz's habilitation discussion (exam) was taken on 11 June 1937 and the questions raised included the following: Zygmund: 1. *Unsolved questions in the theory of trigonometric series, orthogonal series and interpolational polynomials*, 2. *Questions connected with the Laplace–Lyapunov theorem*; Rudnicki: *Integral equations*; Kempisty: *Generalizations of the integral concept.*

After the procedure, on 12 June 1937 his habilitation *On summability of orthogonal series* was approved and the nomination to docent by USB was given. His habilitation lecture had the title *Arithmetization of notion of eventual variable.* The second proposed topic was *Convergence of interpolational polynomials.*

At the age 27 Marcinkiewicz was the youngest doctor with habilitation at the Stefan Batory University. The same year Marcinkiewicz was awarded the Józef Piłsudski Scientific Prize.

[5]Kazimierz Sławiński (1871–1941), chemist, professor of the Stefan Batory University in Wilno.

His senior assistant position at USB was prolonged on the periods 1 September 1937–31 August 1938 and 1 September 1938–31 August 1939. In the meantime, he participated in the 3rd Congress of Polish Mathematicians (29 September–2 October 1937) in Warsaw with the lecture *On one-sided convergence of orthogonal series.* In January 1938 he was nominated for lieutenant reserve, where we can find information about him: height – 180 centimeters, hair – dark blond, eye colour – bright hazel. The same year he received one year scholarship from the Fund for National Culture for the trips to Paris, London and Stockholm to complement his knowledge in probability theory and mathematical statistics. On 9 July 1938 the Ministry of Religious Creeds and Public Education granted Marcinkiewicz paid leave for the academic year 1938/1939.

Photo 10. Józef Marcinkiewicz

On 11 October 1938 Marcinkiewicz presented a talk in Poznań *The development of the probability theory for the last 25 years.* This lecture was probably connected with his application for a professor position in Poznań.

After this visit he went to Paris, where he stayed six months (October 1938–March 1939). In this period Marcinkiewicz collaborated with Stefan Bergman and Raphaël Salem.[6] With Bergman he wrote two joint papers in the theory of complex functions of two variables and with Salem one paper on Riemann sums.

Marcinkiewicz also had contact with the famous mathematician Paul Lévy.[7] Bernard Bru in a conversation with Murad Taqqu discussed about contribution of Louis Bachelier

[6]Stefan Bergman (born 5 May 1895 in Częstochowa – died 6 June 1977 in Palo Alto, California) whose name, in two joint papers with Marcinkiewicz, is written with two "n" at the end, i.e., as Bergmann; Raphaël Salem (born 7 November 1898 in Saloniki – died 20 June 1963 in Paris).

[7]Paul Pierre Lévy (born 15 September 1886 in Paris – died 15 December 1971 in Paris), French mathematician, professor of analysis at École Polytechnique in Paris from 1920 to 1959, who introduced in 1922 the term *functional analysis.* The author of 10 books and over 250 papers in probability, functional analysis and partial differential equations.

(1870–1946) to Brownian motion and informed how Paul Lévy get interested in Brownian motion (see [Ta01]):

> *Lévy began to take an interest in Brownian motion toward the end of the 1930s by way of the Polish school, in particular Marcinkiewicz who was in Paris in 1938. He rediscovered all of Bachelier's results which he had never really seen earlier. Lévy had become enthralled with Brownian motion.*

We should mention here that Marcinkiewicz sent his paper on Brownian motion [M38–40] to the journal already in January 1938 and Lévy published in 1939 paper [Le39] on a problem of Marcinkiewicz. There are also theorems with the Lévy and Marcinkiewicz names in it (cf. our Sections 4.2.3 and 4.6.4).

At the end of 1938 Irena Sławińska arrived in Paris, for a 9 month scientific stay, alumnus of the Polish and Roman literature of USB in Wilno, regarded as his fiancée.[8] She returned to Wilno in August 1939. After the War she was working in Toruń and from 1949 in Lublin (Catholic University of Lublin). I met her in Warsaw on 6 March 2002. She mentioned to me that they were planning to get married and also that:

> *Marcinkiewicz was going out in the middle of the film and was saying that he had no time for entertainment or this is a waste of his time on such a bad movie.*

According to K. Dąbrowski and E. Hensz-Chądzyńska ([DH02], p. 3): *During his stay in Paris he was offered a professorship in one of the American universities. He declined as he had already accepted another offer from the University of Poznań in Poland.* However, I am not aware of any documentary evidence of this American offer.

After Paris Marcinkiewicz arrived in London on his scientific stay. He was staying five months (April–August 1939) at the University College London (UCL). He managed to present his own work in Cambridge, presumably visiting J. E. Littlewood (G. H. Hardy was also there), and in Oxford (cf. [DP10], p. 27). A planned trip to Stockholm was never realized, since Marcinkiewicz returned from London to Wilno at the end of August 1939.

In June 1939 he was appointed Extraordinary Professor at the University of Poznań. From the new academic year 1939/40 Marcinkiewicz should take up the chair of mathematics at the University of Poznań, after Zdzisław Krygowski (1872–1955), who retired in 1938. There is even a protocol of the 8$^{\text{th}}$ meeting of the Senate of the University of Poznań from 23 June 1939, where point 11 is about appointment of docent Marcinkiewicz on associate professor of mathematics and this was presented by the Dean Suszko.[9] Unfortunately, the outbreak of war, disrupted his plans. By the way, Marcinkiewicz on 27 June 1939 was also nominated senior assistant to the mathematics chair at USB for the period 1 September 1939–31 August 1940.

At the end of August 1939 Marcinkiewicz returned to Wilno from London. In the second half of August 1939, Marcinkiewicz stayed in England. The outbreak of war was imminent. In Poland, general mobilisation was announced. Despite his colleagues' advice

[8] Irena Zofia Sławińska (born 30 August 1913 in Wilno – died 18 January 2004 in Warsaw).

[9] Jerzy Suszko (1889–1972), chemist, from 1937 Professor of the University of Poznań.

to stay in England, he decided to go back to Poland. Marcinkiewicz was answering to them that (cf. [MM76], p. 16):

> *as a patriot and son of my homeland would never attempt to refuse the service to the country in such difficult time as war.*

Norman L. Johnson[10] in a conversation with C. B. Read in May 2002 was saying (cf. [Re04], p. 557):

> *I would like to mention the influence that someone had on my life to some extent. In my first, prewar, year on staff at University College, we had a visitor from Poland, Jozef Marcinkiewicz. He was only over for a month or so because he was also visiting Paris. He was a very good theoretical probabilist, he was interested in statistics and he was very remarkable. He was only 28 years old and already a Professor in Poland. We had a lot of talks. I was flattered that he took notice of such a junior member of the staff as I was, in my corner of the laboratory where I felt protected against Fisher. We talked a lot about that, and he came to me for what he called a good practical outlook, thinking that his mathematical statistics ought to be more applicable than it was in the way he had learned it. I also thought I was learning a lot more about mathematics than I had ever learned as an undergraduate in University College. We had a lot of conversations then. When he left in the spring of 1939, it was pretty clear there was going to be a war. He already had been offered a post in the United States, and I said (which almost destroyed our friendship), "Won't you perhaps accept this and be out of the way if the Germans invade Poland?" He was extremely indignant. He said, "My duty is to go back and defend my country, I am a reserve officer and I am surprised you would think of something as bad as that. Why don't you go off to the United States?" I was able to calm him down and sort that out. He did go, and he was taken prisoner by the Russians and ultimately murdered in Katyn Forest near Smolensk. I always felt that I would like to take an opportunity of saying how highly I thought of him as a person and as a probabilist who was appreciative of statistics and that somehow or other thought I could do something useful. At that time when I was just starting, as you know, you are not very sure of anything. I would like to take the opportunity of mentioning that. In fact, in the Encyclopaedia, there is an entry "Marcinkiewicz's Theorem." He wrote a book, but I don't know the title.*

This shows that Marcinkiewicz was not only scientist, but he was also a great Polish patriot returning to Poland. He could have stayed in England or go to United States, but in his opinion, this meant desertion. Instead, he chose to fight and thus became a martyr.

[10] Norman Lloyd Johnson (1917–2004) completed his M.Sc. in statistics in 1938 under supervision of Jerzy Neyman at the Department of Applied Statistics at University College London. The same year he joined the same Department as an Assistant Lecturer on an invitation by Egon Pearson. From 1962 he became Professor at the University of North Carolina at Chapel Hill (USA). He was the author and co-author of 17 books and more than 180 papers in statistics.

Zygmund writes ([Zy64], p. 4):

On September 2nd, the second day of the war, I came across him accidently in the street in Wilno, already in military uniform (he was an officer of the reserve). We agreed to meet the same day in the evening but apparently circumstances prevented him from coming since he did not show up at the appointed place. A few months later came the news that he was a prisoner of the war and was asking for mathematical books. It seems that this was the last news about Marcinkiewicz.

and in [Zy51, p. 8]:

Marcinkiewicz, mobilized, was taken prisoner and disappeared without trace.

Marcinkiewicz was a reserve officer assigned to the 2nd Battalion, 205th Infantry Regiment, and took part in the defence of Lwów (12–21 September 1939). After capitulation of Lwów (22 September 1939) he, together with other officers, was taken prisoner by the Red Army.

Photo 11. Józef Marcinkiewicz in uniform (second from the left).
First from the right – his brother Edward

Stanisława Lewicka (sister of Józef) wrote on 12 October 1959 in the letter to Wiadomości Matematyczne [Mathematical News] ([Le59], pp. 1–2):

During his time in Paris and England, Marcinkiewicz had produced some mathematical work, which he had written down in manuscript form. After returning to Poland he gave these manuscripts to his parents for safe keeping. Sadly Marcinkiewicz's parents suffered the same fate as he did and died in June 1941 in Bukhara (Uzbekistan). After the war his brother Kazimierz

> *accidentally dug this work from the ground, but they were unfortunately in a state of decomposition.*

Stanisław Kolankowski wrote on Józef Marcinkiewicz:

> *We met for the first time on 20 September.* (...) *At night the German army started to leave their positions, and then the Soviet Army came. The Lwów defence committee decided to give the city up to the Soviet Army. The Soviets "temporarily interned" the officers commanding the defence of Lwów.* (...) *It was the 25 of September. I found out that Józef Marcinkiewicz was in the same car with me.* (...) *The railway workers told us we would be located in camps throughout the Soviet Union. Then I decided to flee from the transport along with two other officers from Lwów. I insisted that Marcinkiewicz go with us. He decided not to go.* (...) *The railway workers told us we were going to Starobielsk* (*a small town near Kharkov in Ukraine*)*. Just before the Polish-Soviet border at the Podwysokie station all three of us jumped off the train. I saw Józef Marcinkiewicz at ten p.m. for the last time.*

Marcinkiewicz was kept in the Starobielsk camp from September 1939 until April or May 1940 (registered under the id number 2160; victim index number 6444). The family had received two postcards from the Starobielsk camp. The last one was dated March 1940. Marcinkiewicz also sent some postcards and letters from Starobielsk to his close friends, including Zygmund and Jeśmanowicz.

There exists a description of Marcinkiewicz's stay in the Starobielsk camp, written by Zbigniew Godlewski[11] school colleague, and published in the Review of History (Przegląd Historyczny 38 (1993), z. 2, pp. 323–324) in 1992 *Lived through Starobielsk.*

Probably, the Soviets soon discovered how brilliant their captive was. They offered him some form of collaboration. Marcinkiewicz allegedly asked in a letter for his mathematical books and a copy of his PhD certificate to be sent to him at the camp. It is supposed that, in the end, Marcinkiewicz declined the Soviet offer.

Marcinkiewicz was then murdered in Kharkov, where thousands of Polish officers were executed. He is probably buried in the village Piatykhatky (Piatichatki).

The exact date of Józef Marcinkiewicz's death remains unknown because some official Soviet documents are inaccessible or have been destroyed. The only known information is that this was between 5 April and 12 May 1940.

At the cemetery situated in Janów there is a grave containing the ashes of Kazimierz Marcinkiewicz. A plaque commemorates also Józef Marcinkiewicz and his parents Aleksandra and Klemens Marcinkiewicz. This plaque was founded by the priest Józef Marcinkiewicz in 1956. The plaque bears the names of the parents of Józef, the name of Józef and his brother Kazimierz; above that there are words:

In honour of the martyrdom of the Marcinkiewicz family.

---

[11]Zbigniew Godlewski (1909–1993), from 3 October 1939 prisoner of Starobielsk camp. Memories were written in 1980.

This inscription gives the most tragic fate of the Polish family during the war. Marcinkiewicz's parents Klemens and Aleksandra were transported in June 1941 to Uzbekistan by the NKVD and six months later they died of hunger in Bukhara on 24 December 1941. Józef was executed in Kharkov in Spring 1940. Edward, who was later transported to Siberia, joined the Polish army of General Anders and took part in the battle of Monte Cassino (Italy). He then lived in Argentina, Italy and Switzerland. The youngest brother, Kazimierz, one of the defenders of Lwów, returned to his family's house. Like Mieczysław and Stanisława, he was a member of the Polish underground during the Soviet and Nazi occupation and at the beginning of the Communist regime. In 1946 he was killed by security officers in Janów. Mieczysław was forced by communist authorities to sell the farm and move to a different place (Krapkowice).

Photos 12–13. Symbolic grave of the Marcinkiewicz family in Janów

Józef Marcinkiewicz should be remembered as a true Polish patriot, and especially as an outstanding mathematician. Mathematics was his passion. He possessed an outstanding ability to focus on problems in mathematical thinking and had extraordinary insights in mathematics. Marcinkiewicz's premature death was a huge blow to Polish and world mathematics.

Zygmund, in his article about Marcinkiewicz, wrote ([Zy64], p. 1):

> *His first mathematical paper appeared in 1933*; *the last one he sent for publication in the Summer of 1939. This short period of mathematical activity left, however, a definite imprint on Mathematics, and but for his premature*

death he would probably have been one of the most outstanding contemporary mathematicians. Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.

3. Contests, books, conferences, lectures, exhibitions, awards and special lectures dedicated to Marcinkiewicz. Books, competitions, conferences, exhibitions and awards in his name have since celebrated the memory of Marcinkiewicz. They are presented below in chronological order with a short description.

1947. Poznań. In October 1947 Polish Mathematical Society in Poznań intended to publish *Commemorative Book*, and Dr. Andrzej Alexiewicz (later professor of mathematics) has asked Stanisława Lewicka – the sister of Józef Marcinkiewicz – for biographical material. Unfortunately, the *Book* did not appear in print.

1957. Toruń. In memory of Marcinkiewicz the Toruń Branch of the Polish Mathematical Society initiated in 1957 an annual Marcinkiewicz competition for the best student's mathematical paper. The winners of the first competition in 1957 were: Z. Ciesielski, K. Sieklucki, A. Schinzel and A. Jankowski.

1959. New York. Antoni Zygmund dedicated his famous monograph *Trigonometric Series*, 2nd ed., Cambridge University Press, New York 1959 in the following way: *Dedicated to the memories of A. Rajchman and J. Marcinkiewicz. My teacher and my pupil.*

1960. Warsaw. On the twentieth anniversary of the death of Józef Marcinkiewicz the Mathematical News (Wiadomości Matematyczne) published in Polish an article of Zygmund on Marcinkiewicz: A. Zygmund, *Józef Marcinkiewicz*, Wiadom. Mat. (2) 4 (1960), 11–41.

1964. Warsaw. In recognition of the great mathematical achievements of Marcinkiewicz, Polish Academy of Sciences edited in 1964, on 681 pages, his collected papers: *Józef Marcinkiewicz, Collected Papers*, PWN, Warsaw 1964. Only a few Polish mathematicians have been honoured in this way.

1980. Warsaw. Commission of the History of Mathematics of the Polish Mathematical Society and Institute of History of Science of the Polish Academy of Sciences organized on 11 December 1980 a scientific session dedicated to Józef Marcinkiewicz on the occasion of his 70$^{\text{th}}$ anniversary of birth. Two lectures were given: Z. Ciesielski, *The scientific output of Józef Marcinkiewicz*, L. Jeśmanowicz, *Previous history of the Marcinkiewicz competition*. Moreover, L. Jeśmanowicz was talking about J. Marcinkiewicz.

1981. Chicago. On the occasion of Zygmund's 80$^{\text{th}}$ birthday the conference *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, University of Chicago, Chicago, Ill., March 23–28, 1981 was organized. The wish of Zygmund was that *each speaker should quote or rely on some statement of Marcinkiewicz.* Proceedings of the conference were published in two volumes and edited by William Beckner, Alberto P. Calderón, Robert Fefferman and Peter W. Jones in 1983 on 852 pages.

1981. Olsztyn. The 16$^{\text{th}}$ Scientific Session of the Polish Mathematical Society (15–17 September 1981). Z. Ciesielski presented a talk *Ideas of Józef Marcinkiewicz in mathematical analysis.*

1988. Katowice. Third All Polish School on History of Mathematics *Mathematics at the turn of the twentieth century*, May 1988. Lecture: B. Koszela, *The contribution of Józef Marcinkiewicz, Stefan Mazurkiewicz and Hugo Steinhaus in developing Polish mathematics. A biographical sketch* (in Polish).

1991. Dziwnów. Fifth All Polish School on History of Mathematics *Probability and mechanics in historical sketches*, 9–13 May 1991. Lectures on Marcinkiewicz: E. Hensz and A. Łuczak, *Strong law of large numbers of Marcinkiewicz, classical and non-commutative version*, E. Hensz, *Józef Marcinkiewicz.*

1995. Toruń. Scientific Session of the Polish Mathematical Society (13–15 September 1995). Lecture: K. Dąbrowski and E. Hensz, *Józef Marcinkiewicz* (1910–1940).

2000. Będlewo. 16–20 October 2000. Stefan Banach International Mathematical Center organized *Rajchman–Zygmund–Marcinkiewicz Symposium* dedicated to the memory of Aleksander Rajchman (1891–1940), Antoni Zygmund (1900–1992) and Józef Marcinkiewicz (1910–1940). Talk about Marcinkiewicz: K. Dąbrowski and E. Hensz-Chądzyńska, *Józef Marcinkiewicz* (1910–1940)*. In commemoration of the* 60^{th} *anniversary of his death* ([DH02]).

2007. Gdańsk. Gdańsk Branch of the Polish Mathematical Society and Institute of Mathematics of the Gdańsk University organized on 29 October 2007 an exposition and scientific session to commemorate Józef Marcinkiewicz. Lectures were given by: S. Kwapień, *Józef Marcinkiewicz, Wolfgang Doeblin, two lots, similarities and differences*, Z. Ciesielski, *Some reflections on Józef Marcinkiewicz*, and E. Jakimowicz, *How was the exhibition dedicated to Józef Marcinkiewicz organized.*

2010. Toruń. Scientific session on the hundredth anniversary of birth of Józef Marcinkiewicz (10 March 2010). Lectures: A. Jakubowski, *J. Marcinkiewicz and his achievements in the theory of probability*, Y. Tomilov, *Selected results of J. Marcinkiewicz in the theory of functions and functional analysis.*

2010. Janów. 23 March 2010. *Hundredth anniversary of the birth of Józef Marcinkiewicz* (1910–1940). The School Complex of the Agricultural Education Center in Janów. Lectures: R. Brazis (Polish University, Wilno), *Wilno – a city enlightened in legend of Józef Marcinkiewicz*, L. Maligranda (Luleå University, Sweden), *Józef Marcinkiewicz as pupil, man and mathematician* and exposition of E. Jakimowicz, *To Józef Marcinkiewicz on the occasion of the* 100^{th} *anniversary of his birth and* 70^{th} *anniversary of his death* (continuation of the exposition from 2007).

2010. Iwonicz Zdrój. 25 May 2010. The 24^{th} School of History of Mathematics (24–28 May 2010). Lectures: S. Domoradzki and Z. Pawlikowska-Brożek, *Józef Marcinkiewicz in the light of memories*, L. Maligranda (LTU), *Józef Marcinkiewicz and his mathematical achievements.*

2010. Poznań. 28 June–2 July 2010. On the occasion of the centenary of the birth of Józef Marcinkiewicz (1910–1940), Adam Mickiewicz University (Poznań), Institute of Mathematics of the Polish Academy of Sciences, Warsaw University, Nicolaus Copernicus University (Toruń) organized a scientific conference to commemorate one of the most eminent Polish mathematicians *The Józef Marcinkiewicz Centenary Conference* (JM100). The first plenary lecture was given by L. Maligranda, *Józef Marcinkiewicz* (1910–1940) *– on the centenary of his birth.*

2010. Janów. 14 October 2010. Celebration on which the name of Professor Józef Marcinkiewicz was given to the School Complex of the Agricultural Education Center in Janów [Zespół Szkół Centrum Kształcenia Rolniczego w Janowie, pow. Sokółka].

4. Mathematics of Józef Marcinkiewicz. Results proved by Marcinkiewicz are in the following areas of mathematics:

- Functional Analysis (interpolation of operators, Marcinkiewicz spaces and vector-valued inequalities)
- Probability Theory (independent random variables, Khintchine type inequalities, characteristic functions, Brownian motion)
- Theory of Real Functions
- Trigonometric Series, Power Series, Orthogonal and Fourier Series
- Approximation Theory
- Theory of Functions of Complex Variables.

In the period of six years (1933–1939) Józef Marcinkiewicz wrote 55 papers (while spending one year in the army). 19 were published with co-authors (14 with A. Zygmund, two with S. Bergman and one with S. Kaczmarz, R. Salem, B. Jessen[12] and A. Zygmund). Despite the brevity of his period of mathematical activity, it has nonetheless left a define mark on mathematics.

A list of his published papers can be found in *Józef Marcinkiewicz, Collected Papers*, PWN, Warsaw 1964, pages 31–33, and also here in part 5, which is supplemented by his printed PhD thesis [M35b] and unknown paper [M37b] and is in chronological order.

Marcinkiewicz was also reviewer for *Zentralblatt für Mathematik und ihre Grenzgebiete* (*Zbl*) and *Jahrbuch über die Fortschritte der Mathematik* (*JFM*) in years 1931–1939. He has written 56 reviews.

Marcinkiewicz's papers besides the original and important results, contain a lot of ideas. They are still used today and continue to inspire mathematicians.

Antoni Zygmund has written about his pupil Marcinkiewicz ([Zy64], p. 1):

> *I was one of his professors at the University in Wilno*; *I introduced him to mathematical research and interested him in problems with which I was then concerned. Later on we collaborated and wrote several joint papers*; *but his scientific development was so rapid and the originality of his ideas so great that in certain parts of my own field of work I may only consider myself as his pupil.*

Orlicz asserted that Marcinkiewicz:

> *is probably the only mathematician with whom you can speak about everything. He was a remarkably quick learner.*

Marcinkiewicz was one of the most eminent figures in Polish mathematics and together with Stanisław Zaremba, Leon Lichtenstein, Juliusz Schauder and Antoni Zygmund, the

[12] Borge Christian Jessen (born 19 June 1907 in Copenhagen – died 20 March 1993 in Copenhagen).

most prominent in classical analysis. Zygmund considered him his best pupil, although he had many students in Poland and the USA. We must also remember that it is because of the Master, which was Zygmund, Marcinkiewicz also became a known mathematician.

Photo 14. Wilno, 4 March 1936. Józef Marcinkiewicz (left) and Antoni Zygmund

Paul Nevai in the text of Paul Turán informed ([Tu55], p. 3):

> *Zygmund told me that Marcinkiewicz was the strongest mathematician he ever met – I wonder if I am making this up or he told this to others as well.*

Alberto P. Calderón described ([Ca83], p. xiv):

> *Marcinkiewicz, whose name is familiar to everyone interested in functional analysis and Fourier series, was an extraordinary mathematician. His collaboration with Zygmund lasted almost ten years and produced a number of important results.*

Cora Sadosky, in added information on A. Zygmund and J. Marcinkiewicz, concludes ([Sa01], p. 6):

> *Marcinkiewicz, did became a first-rank mathematician even if he died at 30. Marcinkiewicz is recognized today largely because Zygmund survived the war and became his champion.*

Marcinkiewicz's name in mathematics appeared e.g. in connection to the following: the Marcinkiewicz interpolation theorem, Marcinkiewicz spaces, the Marcinkiewicz integral and Marcinkiewicz function, the Marcinkiewicz–Zygmund inequalities, the Marcinkiewicz–Zygmund law of large numbers, the Marcinkiewicz multiplier theorem, the Jessen–Marcinkiewicz–Zygmund strong differentiation theorem, Marcinkiewicz–Zygmund vector-valued inequalities, the Grünwald–Marcinkiewicz interpolation theorem, the Marcinkiewicz–Salem conjecture, the Marcinkiewicz test for pointwise convergence of Fourier series, the Marcinkiewicz theorem on the Haar system, the Marcinkiewicz theorem on universal primitive function and the Marcinkiewicz theorem on the Perron integral.

A description of Marcinkiewicz's achievements was written in Polish by Antoni Zygmund [Zy60] in 1960 and then translated into English and published in the *Collected Papers* [Zy64] (pages 1–33). Achievements of Marcinkiewicz in analysis were described in Japanese by Satoru Igari [Ig05] (the English translation was published three years later in [Ig08]). We note, moreover, that Philip Holgate delivered on 25 February 1989 a lecture on *Independent functions: probability and analysis in Poland between the wars*, which was published in 1997 (see [Ho97]), and in the third part of this work some important achievements of Marcinkiewicz and Zygmund were discussed.

Also, descriptions of Zygmund achievements have been written by Fefferman, Kahane and Stein [FKS76] and by Stein [St83], which, of course, also contain the discussion of the joint results of Marcinkiewicz and Zygmund.

The article by Zygmund [Zy60] is obviously the best source of Marcinkiewicz's mathematics, however, I have adopted an alternative order of presentation of Marcinkiewicz's results. My order follows appearance of Marcinkiewicz's results in text-books and monographs. This difference is also apparent from the fact that after 50 years, certain sections of mathematics became more popular than others. This is why the first are results in mathematical analysis (in fact in functional analysis), then in probability theory and real analysis to be finished with the classical Fourier series and general orthogonal series, and approximation theory, though Marcinkiewicz began to write papers on Fourier series and approximation theory. This work ends with remarks on some less cited or quoted papers of Marcinkiewicz.

4.1. Functional Analysis. In this section some results are presented that made Marcinkiewicz's name famous in the most spectacular way. These include the Marcinkiewicz interpolation theorem and two types of Marcinkiewicz spaces, as well as Marcinkiewicz–Zygmund vector-valued estimates of operators.

4.1.1. *Marcinkiewicz interpolation theorem* (1939). Consider two classical operators:

(a) The *Hardy operator* H is defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x \in I,$$

where $I = (0,a)$, $0 < a \le \infty$. The operator H is not bounded from $L^1(I)$ to $L^1(I)$ (for example, $f_0(x) = \frac{1}{x\ln^2 x}\chi_{(0,1/2)} \in L^1(0,1)$, but $Hf_0(x) = -1/(x\ln x)$ on $(0,1/2)$ so that $Hf_0 \notin L^1(0,1)$), but it is bounded from $L^1(I)$ to weak-$L^1(I)$ and is bounded from $L^\infty(I)$ to $L^\infty(I)$.

(b) The *maximal operator* M is defined by

$$Mf(x) = \sup_{I\ni x}\frac{1}{|I|}\int_I |f(t)|\,dt, \quad I = (a,b) \subset [0,1].$$

M is not bounded from L^1 to L^1, but is bounded from $L^1(I)$ to weak-$L^1(I)$ and it is bounded from L^∞ to L^∞.

In fact, there are several examples of such type of operators. Marcinkiewicz knowing the Riesz–Thorin interpolation theorem (1926, 1938) and the Kolmogorov result (1925) that the conjugate-function operator is of weak type $(1,1)$ tried to prove theorems not only for the scale of L^p-spaces. A particular version of the Marcinkiewicz interpolation theorem (1939) has the form: *If a linear or sublinear mapping T is of weak type* $(1,1)$ *and strong type* (∞,∞)*, that is, satisfies the estimates*

$$\lambda m\big(\{x \in I : |Tf(x)| > \lambda\}\big) \le A \int_I |f(x)|\,dx \quad \forall \lambda > 0 \tag{1}$$

$$\operatorname*{ess\,sup}_{x\in I} |Tf(x)| \le B \operatorname*{ess\,sup}_{x\in I} |f(x)|, \tag{2}$$

then it is of strong type (p,p) *for* $1 < p \le \infty$*, i.e., we have the estimate*

$$\int_I |Tf(x)|^p\,dx \le C_{A,B} \int_I |f(x)|^p\,dx \tag{3}$$

with

$$C_{A,B}^{1/p} \le 2\Big(\frac{p}{p-1}\Big)^{1/p} A^{1/p} B^{1-1/p}. \tag{4}$$

To understand better the importance of the Marcinkiewicz interpolation theorem let us define the so-called *weak-L^p spaces.* The Lebesgue spaces L^p on the measure space (Ω,Σ,μ) were known for $\Omega = [a,b]$ or $\Omega = \mathbb{R}^n$ already in the thirties for F. Riesz. Investigating the boundedness of operators Marcinkiewicz needed larger spaces in the target, the so-called *weak-L^p spaces* denoted by $L^{p,\infty}$ $(1 \le p < \infty)$ and now called *Marcinkiewicz spaces* given by one of two quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda\mu\big(\{x \in \Omega : |f(x)| > \lambda\}\big)^{1/p}.$$

Note that these spaces are larger than L^p, since for any $\lambda > 0$

$$\int_\Omega |f(x)|^p\,d\mu \ge \int_{\{x\in\Omega : |f(x)|>\lambda\}} |f(x)|^p\,d\mu \ge \lambda^p \mu\big(\{x \in \Omega : |f(x)| > \lambda\}\big),$$

that is,

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \lambda\mu\big(\{x \in \Omega : |f(x)| > \lambda\}\big)^{1/p} \le \Big(\int_\Omega |f(x)|^p\,d\mu\Big)^{1/p} = \|f\|_p.$$

The weak-L^∞ space is here by definition the L^∞ space.

Marcinkiewicz published in 1939 the two-page paper [M39h] and formulated three theorems, containing the one presented below, without proofs. He sent a letter, including the proof of the main theorem for $p_0 = q_0 = 1$ and $p_1 = q_1 = 2$, to Zygmund who after the war reconstructed all the proofs and published them in 1956 (see [Zy56]). This is the reason why Marcinkiewicz's interpolation theorem (1939) is sometimes called the Marcinkiewicz–Zygmund interpolation theorem. Note that in 1953 Zygmund presented all the proofs at his Chicago seminar informing that he was only developing Marcinkiewicz's ideas (cf. [Pe02], p. 46). A proof was also given by the PhD students of Zygmund: Mischa Cotlar for $p_0 = q_0$ and $p_1 = q_1$ (PhD 1953, published in 1956 in [Co56]) and William J. Riordan for $1 \le p_i \le q_i \le \infty$, $i = 0,1$ (PhD 1955, unpublished).

THEOREM 1 (Marcinkiewicz 1939, Zygmund 1956). *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and for* $0 < \theta < 1$ *define* p, q *by the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{5}$$

If $p_0 \le q_0$ *and* $p_1 \le q_1$ (lower triangle) *with* $q_0 \ne q_1$, *then the boundedness of any linear or sublinear operator* $T : L^{p_0} \to L^{q_0,\infty}$ *and* $T : L^{p_1} \to L^{q_1,\infty}$ [*i.e.* T *is of weak type* (p_0, q_0) *and of weak type* (p_1, q_1)] *implies the boundedness of* $T : L^p \to L^q$ [*that is,* T *is of strong type* (p, q)] *and*

$$\|T\|_{L^p\to L^q} \le C\|T\|^{1-\theta}_{L^{p_0}\to L^{q_0,\infty}} \|T\|^{\theta}_{L^{p_1}\to L^{q_1,\infty}}, \tag{6}$$

where

$$\begin{aligned} C = C(\theta, p_0, p_1, q_0, q_1) &= 2\Big(\frac{q}{|q-q_0|} + \frac{q}{|q_1-q|}\Big)^{1/q} \frac{p_0^{(1-\theta)/p_0} p_1^{\theta/p_1}}{p^{1/p}} \\ &= \frac{2\, p_0^{(1-\theta)/p_0} p_1^{\theta/p_1}}{p^{1/p} q^{1/q} [|\frac{1}{q_1} - \frac{1}{q_0}|\, \theta(1-\theta)]^{1/q}} \le 2\Big(\frac{q}{|q-q_0|} + \frac{q}{|q_1-q|}\Big)^{1/q} \\ &= \frac{2}{[|\frac{1}{q_1} - \frac{1}{q_0}| \theta(1-\theta)]^{1/q}}. \end{aligned}$$

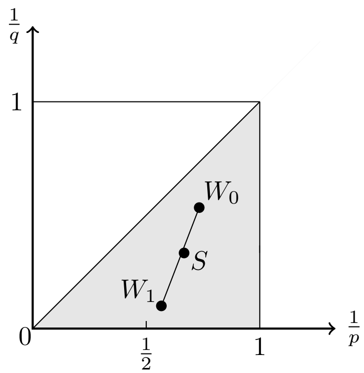

Fig. 1. The Marcinkiewicz interpolation theorem is true in lower triangle: $W_0 = (\frac{1}{p_0}, \frac{1}{q_0})$, $W_1 = (\frac{1}{p_1}, \frac{1}{q_1})$, $S = (\frac{1}{p}, \frac{1}{q})$ and $q_0 \ne q_1$ (except horizontal segments)

Two particular cases of Theorem 1 (A and B) are given below (cf. [JM82], pp. 8–9 and [Gr08], p. 32):

A (Marcinkiewicz's interpolation theorem (diagonal case)). *If* $1 \le p_0 < p_1 \le \infty$ *and* T *is an arbitrary linear or sublinear operator of weak type* (p_0, p_0) *and of weak type* (p_1, p_1), *that is,* $T : L^{p_0} \to L^{p_0,\infty}$ *and* $T : L^{p_1} \to L^{p_1,\infty}$ *is bounded, then it is of strong type* (p, p), *i.e.,* $T : L^p \to L^p$ *is bounded for any* $p_0 < p < p_1$. *Moreover, if* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *for* $0 < \theta < 1$, *then*

$$\|T\|_{L^p\to L^p} \le 2\Big(\frac{p}{p-p_0} + \frac{p}{p_1-p}\Big)^{1/p} \|T\|^{1-\theta}_{L^{p_0}\to L^{p_0,\infty}} \|T\|^{\theta}_{L^{p_1}\to L^{p_1,\infty}}. \tag{7}$$

Remark 1. If the operator T in the above theorem is of weak type (p_0, p_0) and strong type (p_1, p_1), then of course it will be of strong type (p, p), but we obtain then better (than (7)) estimate on the norm, namely the following:

$$\|T\|_{L^p \to L^p} \leq 2 \, \frac{p^{1/p}}{(p - p_0)^{(1-\theta)/p_0} \, p_1^{\theta/p_1}} \, \|T\|^{1-\theta}_{L^{p_0} \to L^{p_0, \infty}} \|T\|^{\theta}_{L^{p_1} \to L^{p_1}} . \tag{8}$$

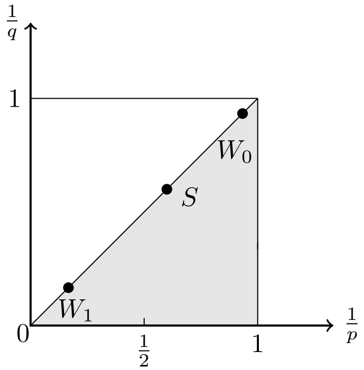

Fig. 2. The Marcinkiewicz interpolation theorem (diagonal case):
$W_0 = (\frac{1}{p_0}, \frac{1}{p_0})$, $W_1 = (\frac{1}{p_1}, \frac{1}{p_1})$, $S = (\frac{1}{p}, \frac{1}{p})$

B (Little Marcinkiewicz interpolation theorem). *If $1 < p \leq \infty$ and T is an arbitrary linear or sublinear operator of weak type $(1, 1)$ and strong type (∞, ∞), that is, $T : L^1 \to L^{1,\infty}$ and $T : L^\infty \to L^\infty$ is bounded, then it is of strong type (p, p), i.e., $T : L^p \to L^p$ is bounded and*

$$\|T\|_{L^p \to L^p} \leq 2 \Big(\frac{p}{p-1} \Big)^{1/p} \|T\|^{1/p}_{L^1 \to L^{1,\infty}} \|T\|^{1-1/p}_{L^\infty \to L^\infty} . \tag{9}$$

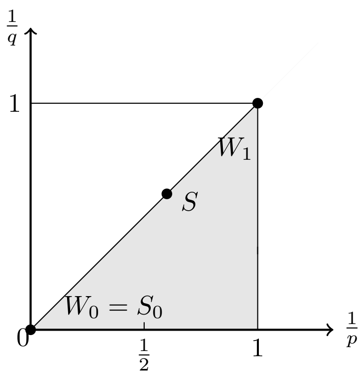

Fig. 3. Little Marcinkiewicz interpolation theorem:
$W_0 = S_0 = (0, 0)$, $W_1 = (1, 1)$, $S = (\frac{1}{p}, \frac{1}{p})$

A natural question appears, namely if the reverse theorem is true, but there are linear and sublinear bounded operators in L^p for all $1 < p \leq \infty$, which are not of weak type $(1, 1)$. For example, such an operator is the two- or more-dimensional strong maximal operator (averaging and supremum are taken over rectangles) as sublinear operator, and as a linear operator we can take arbitrary linear operator majorized by this sublinear operator. In connection to the Marcinkiewicz interpolation theorem we formulate several remarks:

1. Marcinkiewicz was probably the first who used the word "interpolation of operators". Riesz and Thorin spoke on "convexity theorems" (see also Peetre [Pe02], p. 39 and Horvath [Ho09], p. 618). Marcinkiewicz's proof is based on an idea of decomposition of a function which generates later on the concept of the K-functional playing a central role in modern interpolation theory.

2. It is not true, as some authors write, that Marcinkiewicz obtained in his short paper the result only in the diagonal case. Marcinkiewicz had the theorem in the general case and equations (5) were written by formulas $p_0 < p < p_1$ and $\frac{q-q_0}{q_1-q} = \frac{q_0 p_1}{p_0 q_1} \cdot \frac{p-p_0}{p_1-p}$. Moreover, Marcinkiewicz's second theorem was formulated even for Orlicz spaces (in this case it was indeed the diagonal case): if a linear or sublinear operator is bounded $T : L^{p_0} \to L^{p_0}$ and $T : L^{p_1} \to L^{p_1}$, and a function $\varphi : [0, \infty) \to [0, \infty)$ is continuous, increasing, vanishing at zero and satisfying three conditions $\varphi(2u) = O(\varphi(u))$, $\int_1^u t^{-p_0-1}\varphi(t)\, dt = O(u^{-p_0}\varphi(u))$ and $\int_u^\infty t^{p_1-1}\varphi(t)\, dt = O(u^{-p_1}\varphi(u))$ as $u \to \infty$, then for f such that $\varphi(|f|) \in L^1[0, 1]$ we obtain $\int_0^1 \varphi(|Tf(x)|)\, dx \le C \int_0^1 \varphi(|f(x)|)\, dx + C$, where C is independent of f (see Zygmund [Zy56], Theorem 2 and [Zy59], XII. Theorem 4.22).

3. Marcinkiewicz's interpolation theorem was proved even for pointwise quasi-additive operators, that is, if there exists a constant $\gamma \ge 1$ such that for any measurable functions f, g we have the following inequality μ-almost everywhere on Ω:

$$|T(f + g)(x)| \le \gamma\big(|Tf(x)| + |Tg(x)|\big).$$

4. Marcinkiewicz's interpolation theorem is true, if $p \le q$ and $q_0 \ne q_1$, that is, one point can be in the upper triangle but the point S must appear in the lower triangle. Proof of this theorem was given by: Calderón (1963), Lions–Peetre (1964), O'Neil (1964), Hunt (1964) and Krée (1967). Berenstein–Cotlar–Kerzman–Krée (1967) proved that if for the segment W_0W_1 Marcinkiewicz's interpolation theorem is true, then it is also true for another segment obtained from the rotation of W_0W_1 around the point S (except horizontal and vertical segments).

5. Marcinkiewicz's interpolation theorem is NOT true in the upper triangle, that is, if $p_0 > q_0$ and $p_1 > q_1$. A counter-example is due to Hunt (1964). He also observed that Theorem 1 is true in the extended range $0 < p_0, p_1, q_0, q_1 \le \infty$ provided $p \le q$ and $q_0 \ne q_1$ (cf. Hunt [Hu64], p. 807).

6. E. Stein and G. Weiss (1959) generalized the Marcinkiewicz interpolation theorem replacing the spaces L^{p_i} in the domain of an operator by the smaller Lorentz spaces $L^{p_i,1}$, $i = 0, 1$ (in fact, they have in the assumption of the Marcinkiewicz theorem only estimates for characteristic functions of measurable sets). Hence, if $p_i \le q_i$, $p_i \ne \infty$, $i = 0, 1$ and $q_0 \ne q_1$, then the boundedness of $T : L^{p_0,1} \to L^{q_0,\infty}$ and $T : L^{p_1,1} \to L^{q_1,\infty}$ implies the boundedness of $T : L^p \to L^q$.

7. Using reiteration theorems for the real method of interpolation we are getting a *generalized Marcinkiewicz interpolation theorem*: *Suppose* $1 \le p_0, p_1 < \infty$, $1 \le q_0, q_1 \le \infty$ *with* $q_0 \ne q_1$. *If a quasi-linear operator* $T : L^{p_0,1} \to L^{q_0,\infty}$ *and* $T : L^{p_1,1} \to L^{q_1,\infty}$ *is*

bounded, then $T : L^{p,r} \to L^{q,r}$ *is bounded for any* $1 \le r \le \infty$*. In particular,* $T : L^p \to L^{q,p}$ *is bounded.*

If we have, as in Marcinkiewicz's interpolation theorem, $p_0 \le q_0$ and $p_1 \le q_1$, then $L^{q,p} \hookrightarrow L^q$.

8. A very important progression to a generalization of the Marcinkiewicz interpolation theorem was done by Calderón (1966), who found the maximal operator in the sense that if an operator $T : L^{p_0,1} \to L^{q_0,\infty}$ and $T : L^{p_1,1} \to L^{q_1,\infty}$ is bounded, then $(Tf)^*(t) \le CS_\sigma(f^*)(t)$ for all $t > 0$, where S_σ is the maximal Calderón operator:

$$S_\sigma f(t) = \int_0^\infty f(s) \min\Big\{\frac{s^{1/p_0}}{t^{1/q_0}}, \frac{s^{1/p_1}}{t^{1/q_1}}\Big\} \frac{ds}{s}$$
$$= t^{-1/q_0} \int_0^{t^m} s^{1/p_0 - 1} f(s)\, ds + t^{-1/q_1} \int_{t^m}^\infty s^{1/p_1 - 1} f(s)\, ds,$$

with $m = (1/q_0 - 1/q_1)/(1/p_0 - 1/p_1)$. To get the Marcinkiewicz interpolation theorem it is enough to investigate boundedness of the last two operators of Hardy type.

9. Marcinkiewicz's interpolation theorem was proved for symmetric spaces by Boyd (1967 $\Rightarrow$, 1969 $\Leftrightarrow$): if $1 \le p_0 < p_1 < \infty$, E is a symmetric space with the Fatou property of the norm on either $I = [0,1]$ or $I = [0,\infty)$ and every linear operator $T : L^{p_0,1} \to L^{p_0,\infty}$ and $T : L^{p_1,1} \to L^{p_1,\infty}$ is bounded, then it yields that $T : E \to E$ is bounded if and only if $1/p_1 < \alpha_E \le \beta_E < 1/p_0$, where numbers α_E, β_E are so-called Boyd indices of the space E defined by

$$\alpha_E = \lim_{a\to 0^+} \frac{\ln \|\sigma_a\|_{E\to E}}{\ln a}, \quad \beta_E = \lim_{a\to\infty} \frac{\ln \|\sigma_a\|_{E\to E}}{\ln a}$$

and $\sigma_a f(x) = f(x/a)\chi_I(x/a)$. Krein–Petunin–Semenov (1977) proved that Boyd's theorem is true for arbitrary symmetric spaces (even without the Fatou property of the norm). In particular, E is an interpolation space between L^{p_0} and L^{p_1}. If $p_1 = \infty$, a one-sided estimate for a symmetric space E, $\beta_E < 1/p_0$, $1 \le p_0 < \infty$, implies that E is an interpolation space between L^{p_0} and L^∞ (see Maligranda [Mal81], Theorem 4.6, where it is proved even for Lipschitz operators). Moreover, Astashkin and Maligranda [AM04] proved the following one-sided Boyd theorem: if a symmetric space E either has the Fatou property or is separable and $\alpha_E > 1/p_1$, $1 < p_1 < \infty$, then E is an interpolation space between L^1 and L^{p_1}.

10. Marcinkiewicz's interpolation theorem is not true for bilinear operators without additional assumptions. In fact, Strichartz (1969) proved the following: the operator

$$S(f,g)(x) = \int_0^\infty f(xt)g(t)\, dt$$

is bounded $S : L^1 \times L^\infty \to L^{1,\infty}$ and $S : L^2 \times L^2 \to L^{2,\infty}$, but it is not bounded $S : L^p \times L^{p'} \to L^p$, and Maligranda (1989) proved that the operator

$$T(f,g)(x) = \int_0^1 \int_0^1 f(s)g(t) \min\Big(\frac{1}{st}, \frac{1}{x}\Big)\, ds\, dt$$

is bounded $T : L^1 \times L^1 \to L^{1,\infty}$ and $T : L^2 \times L^2 \to L^{2,\infty}$, but it is not bounded $T : L^p \times L^p \to L^p$ for $1 < p \le 2$.

J.-L. Lions and J. Peetre (1964) proved that for the real method of interpolation we have the following interpolation theorem for bilinear operators: if a bilinear operator $T : L^{p_0} \times L^{q_0} \to L^{r_0,\infty}$ and $T : L^{p_1} \times L^{q_1} \to L^{r_1,\infty}$ is bounded, then $T : L^p \times L^q \to L^r$ is bounded, if besides the natural interpolation equality

$$\Big(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\Big) = (1-\theta)\Big(\frac{1}{p_0}, \frac{1}{q_0}, \frac{1}{r_0}\Big) + \theta\Big(\frac{1}{p_1}, \frac{1}{q_1}, \frac{1}{r_1}\Big)$$

we have also $1/r \le 1/p + 1/q - 1$. More theorems of this type can be found in papers by Sharpley (1977), Zafran (1978), Janson (1988) and Grafakos–Kalton (2001).

11. Marcinkiewicz's interpolation theorem for spaces of sequences (and some of its analogues) was given by Sargent in 1961.

12. In 1972 Yoram Sagher (cf. [Sa72], p. 172 and [Sag72], p. 240) introduced the notion of a *Marcinkiewicz quasi-cone*. If (A_0, A_1) is a pair of quasi-normed spaces, then a subset Q of $A_0 + A_1$ is called a quasi-cone if $Q + Q \subset Q$. Q is a cone if we also have $\lambda Q \subset Q$ for all $\lambda > 0$. A quasi-cone Q is called a *Marcinkiewicz quasi-cone* in (A_0, A_1) if

$$(A_0 \cap Q, A_1 \cap Q)_{\theta,p} = (A_0, A_1)_{\theta,p} \cap Q \quad \text{for all } 0 < \theta < 1,\ 0 < p \le \infty,$$

where $(\cdot,\cdot)_{\theta,p}$ means the real K-method of interpolation of Lions-Peetre. For example, $Q = \{(x_k)_{k=1}^{\infty} : x_k \downarrow 0\}$ is a Marcinkiewicz quasi-cone in (l^p, l^∞).

13. In 1978 Dmitriev and Krein [DK78] extended Marcinkiewicz interpolation theorem to operators mapping couple of Banach spaces (A_0, A_1) into couple of Marcinkiewicz spaces $(M^*_{\varphi_0}, M^*_{\varphi_1})$.

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[Zy56] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operations*, J. Math. Pures Appl. (9) 35 (1956), 223–248.

4.1.2. *Marcinkiewicz function and sequence spaces* (1939). Marcinkiewicz's investigations led him to consider three types of spaces: two symmetric spaces and one space of another type, which will be considered in the next part. All are called now *Marcinkiewicz spaces*. Earlier we discussed about the weak-L^p space $L^{p,\infty}$ $(1 \le p < \infty)$ or *Marcinkiewicz space* given by one of quasi-norms:

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) = \sup_{\lambda>0} \lambda\mu(\{x \in \Omega : |f(x)| > \lambda\})^{1/p}.$$

The above type of space can be easily generalized. Let $I = (0,1)$ or $I = (0,\infty)$ and let $\varphi : I \cup \{0\} \to [0,\infty)$ be an arbitrary concave function on I such that $\varphi(0) = 0$ (it is also possible to take as φ only quasi-concave function, that is, a function for which inequality $\varphi(s) \le \max(1, s/t)\varphi(t)$ is true for all $s, t \in I$). The *Marcinkiewicz function space* M_φ^* on I contains classes of all measurable functions generated by the quasi-norm

$$\|f\|_\varphi^* = \sup_{t\in I} \varphi(t)\, f^*(t) < \infty,$$

where f^* denotes the decreasing rearrangement of $|f|$.

Important is also another (smaller) *Marcinkiewicz function space* M_φ on I generated by the norm

$$\|f\|_\varphi = \sup_{t\in I} \varphi(t)\, f^{**}(t), \quad \text{where } f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

In the case when $\varphi(t) = t^{1/p}$, $1 < p < \infty$ we have $M_\varphi^* = M_\varphi = L^{p,\infty}$, but for $\varphi(t) = t$ we get $M_\varphi^* = L^{1,\infty}$ (weak-L^1 space is a quasi-Banach space but not a Banach space, since the triangle inequality holds with constant 2 and $M_\varphi = L^1$).

Of course, $M_\varphi \subset M_\varphi^*$ and $\|f\|_\varphi^* \le \|f\|_\varphi$ for $f \in M_\varphi$. The Marcinkiewicz function space M_φ on I is a symmetric Banach space and for an arbitrary symmetric space X on I with the fundamental function $\varphi(t) := \|\chi_{(0,t)}\|_X$, M_φ is the largest symmetric space containing X with the same fundamental function, i.e, $\|\chi_{(0,t)}\|_X = \|\chi_{(0,t)}\|_{M_\varphi} = \varphi(t)$ for any $t \in I$. This follows from the fact that $\int_0^t f^*(s)\, ds \le \frac{t}{\varphi(t)}\|f^*\chi_{(0,t)}\|_X$ for $f \in X$.

Let us recall that a symmetric space X on I is an ideal Banach space on I (the assumption $|f(t)| \le |g(t)|$ almost everywhere on I, $g \in X$ and f is measurable on I implies that $f \in X$ and $\|f\|_X \le \|g\|_X$) with the additional property that two arbitrary equimeasurable functions f and g, i.e. satisfying $m(\{x \in I : |f(x)| > \lambda\}) = m(\{x \in I : |g(x)| > \lambda\})$ for any $\lambda > 0$, with $f \in X$ and g measurable on I gives $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$.

Note that when we investigate the Marcinkiewicz spaces $L^{p,\infty}$ or operators with values in this space, then important is the so-called Kolmogorov–Cotlar equivalence (cf. García-Cuerva and Rubio de Francia [GR85], pp. 485–486): if $0 < q < p < \infty$ and $f \in L^{p,\infty}$, then

$$\|f\|_{p,\infty} \approx \sup_{A \subset I, 0<\mu(A)<\infty} \mu(A)^{1/p-1/q} \Big(\int_A |f(x)|^q \, d\mu\Big)^{1/q}. \tag{10}$$

Marcinkiewicz sequence spaces m_φ^* and m_φ are defined analogously by quasi-norms and norms

$$\|x\|_\varphi^* = \sup_{n\in\mathbb{N}} \varphi(n) x_n^* < \infty, \quad \|x\|_\varphi = \sup_{n\in\mathbb{N}} \varphi(n) \,\frac{1}{n} \sum_{k=1}^{n} x_k^*\,,$$

where $\varphi : \mathbb{N} \to [0,\infty)$ satisfies $\varphi(k) \le \max(1, k/n)\varphi(n)$ for any $k, n \in \mathbb{N}$ and $({x_n}^*)$ is a rearrangement of the sequence $(|x_n|)$ in decreasing order. These spaces are also symmetric sequence spaces.

Marcinkiewicz function and sequence spaces (symmetric) are now classical spaces and are still investigated (some examples are given below). They also appear naturally in the interpolation theory.

Marcinkiewicz symmetric spaces, as examples or objects of investigation of their structure, can be found e.g. in the following 5 monographs:

[KJF77] A. Kufner, O. John, S. Fučik, *Function Spaces*, Academia, Prague 1977 [4.2. Marcinkiewicz spaces and their connection with the spaces $L_p^w(\Omega)$, pp. 209–212].

[PP80] Ju. I. Petunin, A. N. Plichko, *The Theory of the Characteristics of Subspaces and its Applications*, Vishcha Shkola, Kiev, 1980 (Russian) [Marcinkiewicz spaces M_ψ and M_ψ^0, pp. 101–103 and 111–112].

[KPS82] S. G. Krein, Yu. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian); English transl.: Transl. Math. Monogr. 54, Amer. Math. Soc., Providence, 1982 [II.5, p. 107: Lorentz and Marcinkiewicz spaces; Marcinkiewicz spaces, pp. 112–118; II.6.2. Operators from Lorentz spaces to Marcinkiewicz spaces, pp. 127–130].

[AZ90] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts Math. 95, Cambridge Univ. Press, Cambridge, 1990 [5.2. Lorentz and Marcinkiewicz spaces].

[BK91] Yu. A. Brudnyĭ, N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces I*, North-Holland, Amsterdam, 1991 [The Marcinkiewicz space $M(\varphi)$, p. 472].

The name *Marcinkiewicz space* appeared also in *Encyclopaedia of Mathematics*:

[Kr90] S. G. Kreĭn, *Marcinkiewicz space*, in: Encyclopaedia of Mathematics, Vol. 6, Kluwer, Dordrecht, 1990, 93–94.

Papers having word *Marcinkiewicz space* in the title are for instance:

[AK08] M. D. Acosta, A. Kamińska, *Norm-attaining operators between Marcinkiewicz and Lorentz spaces*, Bull. Lond. Math. Soc. 40 (2008), 581–592.

[AS92] J. Appell, E. M. Semenov, *On the equivalence of the Lorentz and Marcinkiewicz norm on subsets of measurable functions*, J. Funct. Anal. 104 (1992), 47–53.

[AS95] J. Appell, E. M. Semenov, *On the truncation of functions in Lorentz and Marcinkiewicz spaces*, Rocky Mountain J. Math. 25 (1995), 857–866.

[As07] S. V. Astashkin, *On the normability of Marcinkiewicz classes*, Mat. Zametki 81 (2007), 483–489; English transl.: Math. Notes 81 (2007), 429–434.

[AL06] S. V. Astashkin, K. V. Lykov, *Extrapolation description of Lorentz and Marcinkiewicz spaces "close" to L_∞*, Sibirsk. Mat. Zh. 47 (2006), 974–992; English transl.: Siberian Math. J. 47 (2006), 797–812.

[AS07] S. V. Astashkin, F. A. Sukochev, *Banach–Saks property in Marcinkiewicz spaces*, J. Math. Anal. Appl. 336 (2007), 1231–1258.

[BL10] C. Boyd, S. Lassalle, *Geometry and analytic boundaries of Marcinkiewicz sequence spaces*, Q. J. Math. 61 (2010), 183–197.

[CN85] M. Cwikel, P. Nilsson, *Interpolation of Marcinkiewicz spaces*, Math. Scand. 56 (1985), 29–42.

[KK05] N. J. Kalton, A. Kamińska, *Type and order convexity of Marcinkiewicz and Lorentz spaces and applications*, Glasg. Math. J. 47 (2005), 123–137.

[KL04] A. Kamińska, H. J. Lee, *M-ideal properties in Marcinkiewicz spaces*, Comment. Math. Prace Mat. 2004, Tomus specialis in Honorem Juliani Musielak, 123–144.

[KLL09] A. Kamińska, H. J. Lee, G. Lewicki, *Extreme and smooth points in Lorentz and Marcinkiewicz spaces with applications to contractive projections*, Rocky Mountain J. Math. 39 (2009), 1533–1572.

[KP08] A. Kamińska, A. M. Parrish, *Convexity and concavity constants in Lorentz and Marcinkiewicz spaces*, J. Math. Anal. Appl. 343 (2008), 337–351.

[KP10] A. Kamińska, A. M. Parrish, *Note on extreme points in Marcinkiewicz function spaces*, Banach J. Math. Anal. 4 (2010), 1–12.

[Lo75] G. Ja. Lozanovskiĭ, *Coordinate Marcinkiewicz spaces*, Optimizacija 17 (34) (1975), 130–142 (Russian).

[Lo78] G. Ja. Lozanovskiĭ, *The representation of linear functionals in Marcinkiewicz spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. 1978, no. 1 (188), 43–53 (Russian).

[Me75] A. A. Mekler, *The Hardy–Littlewood property in Marcinkiewicz spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. 1975, no. 3 (154), 104–106 (Russian).

[Mi78] M. Milman, *Embeddings of Lorentz–Marcinkiewicz spaces with mixed norms*, Anal. Math. 4 (1978), 215–223.

[Pu01] E. Pustylnik, *On some properties of generalized Marcinkiewicz spaces*, Studia Math. 144 (2001), 227–243.

[Se09] A. A. Sedaev, *Singular symmetric functionals and stabilizing subsets of the Marcinkiewicz space*, Izv. Vyssh. Uchebn. Zaved. Mat. 2009, no. 12, 90–94; English transl.: Russian Math. (Iz. VUZ) 53 (2009), no. 12, 77–80.

[To78] E. V. Tokarev, *Quotient spaces of Marcinkiewicz spaces*, Sibirsk. Mat. Zh. 19 (1978), 704–707; English transl.: Siberian Math. J. 19 (1978), 498–500.

[To84] E. V. Tokarev, *Quotient spaces of Banach lattices and Marcinkiewicz spaces*, Sibirsk. Mat. Zh. 25 (1984), 205–212; English transl.: Siberian Math. J. 25 (1984), 332–338.

4.1.3. *Marcinkiewicz $M^p(\mathbb{R})$ spaces* (1939). The third type of Marcinkiewicz function spaces (sometimes also called Besicovitch–Marcinkiewicz spaces) consists of functions defined on $\mathbb{R}$ satisfying certain condition, more precisely, if $1 \leq p < \infty$, then the *Marcin-*

kiewicz space $M^p = M^p(\mathbb{R})$ contains all measurable real (or complex) functions $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) locally p-integrable on $\mathbb{R}$ and such that the semi-norm

$$\|f\|_{M^p} = \limsup_{T\to\infty}\Big(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt\Big)^{1/p} < \infty. \tag{11}$$

The closure in the semi-norm $\|\cdot\|_{M^p}$ of the set of trigonometric polynomials $\sum a_k e^{i\lambda_k t}$ is the Besicovitch space of almost periodic functions (B^p a.p.). Marcinkiewicz himself was calling this space the Besicovitch space and he proved the completeness of this space in the paper [M39b].

THEOREM 2 (Marcinkiewicz 1939). *The space* $M^p(\mathbb{R})$ *with semi-norm* (11) *is complete.*

Marcinkiewicz and Orlicz were trying to prove completeness in Lwów, when Marcinkiewicz was there in the academic year 1935/1936, but they were not able to do it. Two years later, on 24 November 1938, Marcinkiewicz wrote in a letter from Paris to Orlicz the proof of completeness:

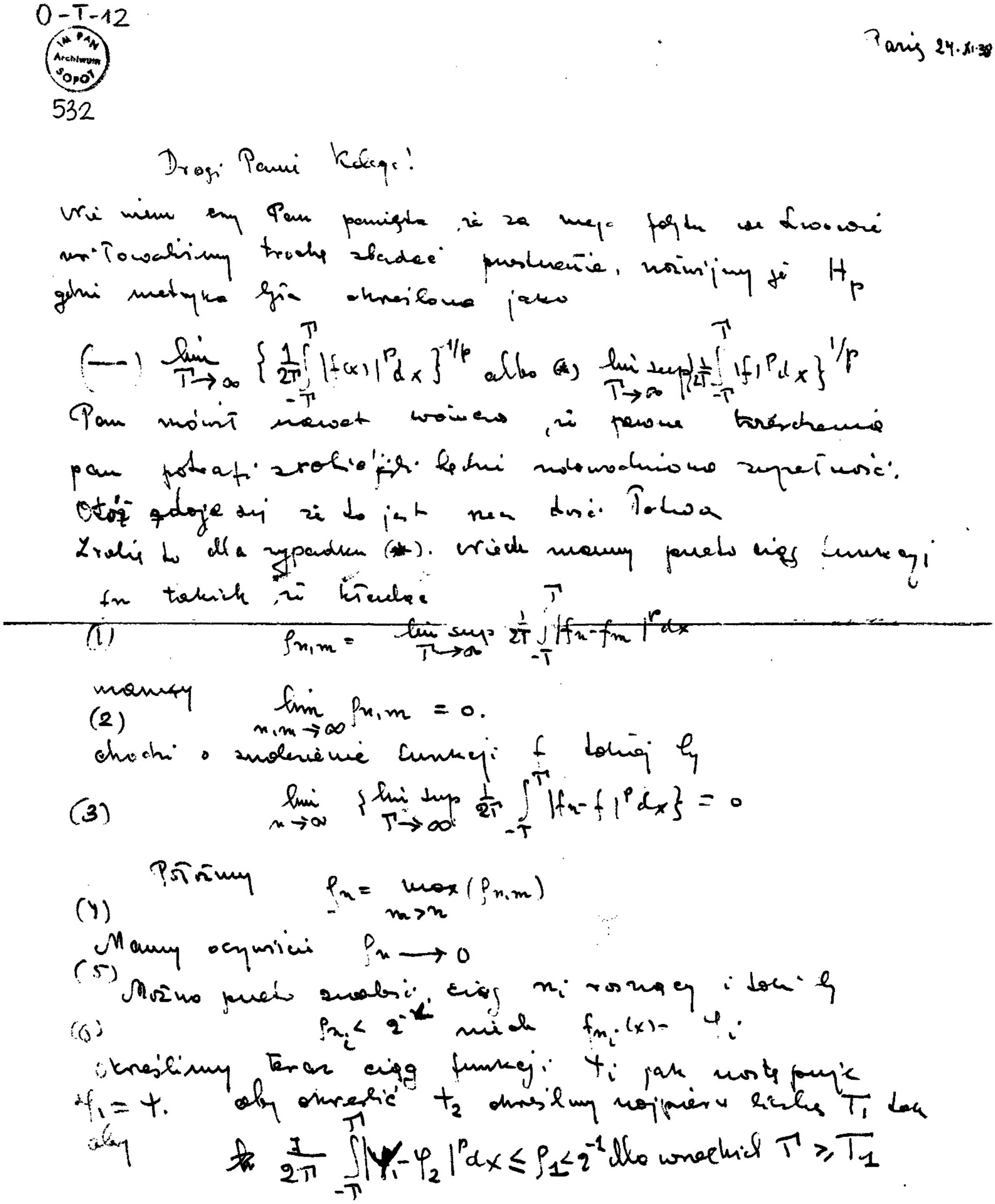
> *Dear Colleague, Sir!*
>
> *I don't know if You remember that at the time of my stay in Lwów we were trying to investigate spaces, let say* H_p*, where the metric is defined as*
>
> $$(-)\ \lim_{T\to\infty}\Big\{\frac{1}{2T}\int_{-T}^{T}|f(x)|^p\,dx\Big\}^{1/p} \quad \text{or} \quad (*)\ \limsup_{T\to\infty}\Big\{\frac{1}{2T}\int_{-T}^{T}|f(x)|^p\,dx\Big\}^{1/p}.$$
>
> *You were saying at that time that you can prove some theorems if we have completeness. I think that this is quite simple. I will do it in the case* $(*)$*.*
> (...) *These spaces however will be different from spaces* L^p*, since they are* <u>*non-separable*</u>*. Therefore results which we can get may be different from those in* L^p*. I would like to work with you in their investigations, but unfortunately I am doing several different small things taking time and I think you will not have a great benefit from me.*
> *In Paris I feel great. Local mathematicians are so-so, but there are many foreigners and it is possible to have discussions about many things. Besides it is a free time from classes and, finally, it is an interesting city in terms of general culture. Best regards, signed by Marcinkiewicz.*

Note here that completeness of the space where "lim sup" is replaced by "sup", in the terms of a new convergence, was given already in 1914 by the Italian mathematician Pia Nalli (1886–1964). Marcinkiewicz gave a proof of completeness for $(*)$ in [M39b] (see also Levitan [Le53], pp. 249–252), and other proofs were also found by Bohr and Følner [BF45, pp. 54–57], Hartman and Wintner [HW47], Luxemburg and Zaanen [LZ63], and Corduneanu [Co09].

Observe that if $1 \le p \le q < \infty$, then

$$L^\infty(\mathbb{R}) \overset{1}{\hookrightarrow} M^q(\mathbb{R}) \overset{1}{\hookrightarrow} M^p(\mathbb{R}),$$

that is, $L^\infty(\mathbb{R}) \subset M^q(\mathbb{R}) \subset M^p(\mathbb{R})$ and $\|f\|_{M^p} \le \|f\|_{M^q}$ for any $f \in M^q(\mathbb{R})$ and $\|f\|_{M^q} \le \|f\|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R})$.



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Paris 24.XI.38

Drogi Panie Kolego!

Nie wiem czy Pan pamięta, że za mego pobytu we Lwowie rozmawialiśmy trochę [illegible] przestrzenie, oznaczmy je H_p gdzie metryka jest określona jako

$$(-)\ \lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|f(x)|^p\,dx\right\}^{1/p} \text{ albo } (*)\ \limsup_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}|f|^p\,dx\right\}^{1/p}$$

Pan mówił nawet wówczas, że pewne twierdzenia pan potrafi [illegible] udowodnioną zupełność. Otóż zdaje mi się że to jest nawet dość łatwo.

Zrobię to dla wypadku (*). Niech mamy przeto ciąg funkcji f_n takich, że kładąc

$$(1)\quad \rho_{n,m} = \limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_n-f_m|^p\,dx$$

mamy

$$(2)\quad \lim_{n,m\to\infty}\rho_{n,m} = 0.$$

Chodzi o znalezienie funkcji f takiej by

$$(3)\quad \lim_{n\to\infty}\left\{\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_n-f|^p\,dx\right\} = 0$$

Połóżmy

$$(4)\quad \rho_n = \max_{m>n}(\rho_{n,m})$$

Mamy oczywiście

$$(5)\quad \rho_n \to 0$$

Można przeto znaleźć ciąg n_i rosnący i taki by

$$(6)\quad \rho_{n_i} < 2^{-i} \quad \text{niech } f_{n_i}(x) = \varphi_i$$

Określimy teraz ciąg funkcji ψ_i jak następuje $\psi_1 = \varphi_1$. aby określić ψ_2 określimy najpierw liczbę T_1 tak aby

$$\frac{1}{2T}\int_{-T}^{T}|\psi_1-\varphi_2|^p\,dx \le \rho_1 < 2^{-1} \text{ dla wszelkich } T \geq T_1$$

Photo 15. Paris, 24 Nov. 1938. The fascimile of J. Marcinkiewicz's letter to W. Orlicz (p. 1)

Sometimes the name *Marcinkiewicz space* is used on the quotient space $\mathcal{M}^p = \mathcal{M}^p(\mathbb{R})$, i.e. the space $M^p(\mathbb{R})$ modulo the kernel of $\|\cdot\|_{M^p}$ (that is, $\|\cdot\|_{M^p} = 0$). This space is then a Banach ideal space on $\mathbb{R}$.

Lau (1980) investigated geometry of the ball and geometry of the Marcinkiewicz space. He proved, among other things, the following theorem:

The Marcinkiewicz space $\mathcal{M}_p(\mathbb{R})$ contains an isomorphic copy of l^∞. Therefore, $\mathcal{M}_p(\mathbb{R})$ is a non-separable and non-reflexive space.

The convolution operator was investigated by Bertrandias (1966), who proved that if μ is a measure on $\mathbb{R}$ and $f \in M^p$, then the convolution $f*\mu \in M^p$ and $\|f*\mu\|_p \le \|f\|_p\|\mu\|$. While Lau (1981) proved that if $M_r^p = \{f \in M^p : \lim_{T\to\pm\infty} \frac{1}{T}\int_T^{T+1} |f(t)|^p\,dt = 0\}$, then for the measure μ the norm of the convolution operator on M_r^p is equal to the norm of the convolution on $L^p(\mathbb{R})$.

Note that there are generalizations of these spaces, the so-called *Marcinkiewicz–Orlicz spaces* generated by modulars $\rho_\varphi(f) = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(|f(t)|)\,dt$, where $\varphi : [0,\infty) \to [0,\infty)$ are convex Orlicz functions. These spaces were developed by Albrycht (1956, 1959 and 1962), Wang (1959) and Hillmann (1986). In 1995 Kucher and Plichko [KP95], [KPl95] investigated also the *Marcinkiewicz-symmetric spaces* M_E defined by semi-norms $\|f\|_{M_E} = \limsup_{T\to\infty} \|f_T\|_E$, where $f_T(t) = f(tT)$ and E is an arbitrary symmetric Banach function space on $[-\frac{1}{2},\frac{1}{2}]$. Moreover, Vo–Khac in [BCD87] and Cohen–Losert in [CL06] defined *generalized Marcinkiewicz spaces* based upon arbitrary measure spaces and limits of averages over more general families of sets.

Harmonic analysis in Marcinkiewicz spaces was investigated in [Ur61], [Be66] and [BCD87].

Marcinkiewicz spaces $M^p(\mathbb{R})$ are considered in the following books and papers:

[Le53] B. M. Levitan, *Almost-Periodic Functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953 [5.10.1. Marcinkiewicz theorem, pp. 249–252].

[Ba84] J. Bass, *Fonctions de corrélation, fonctions pseudo-aléatoires et applications*, Masson, Paris, 1984 [Marcinkiewicz space M^2, pp. 33–35].

[Pa85] A. A. Pankov, *Bounded and Almost Periodic Solutions of Nonlinear Operator-Differential Equations*, Naukova Dumka, Kiev, 1985 [Marcinkiewicz, pp. 41, 179]; English transl.: Kluwer, Dordrecht, 1990 [Marcinkiewicz, pp. 13, 43, 220].

[BCD87] J.-P. Bertrandias, J. Couot, J. Dhombres, M. Mendès France, Pham Phu Hien, Kh. Vo-Khac, *Espaces de Marcinkiewicz: corrélations, mesures, systèmes dynamiques*, Masson, Paris, 1987 [Marcinkiewicz spaces, pp. 2–3, 11, 13, 59, 78].

[Co09] C. Corduneanu, *Almost Periodic Oscillations and Waves*, Springer, New York, 2009 [Marcinkiewicz space, pp. 41–46].

[Al56] J. Albrycht, *Some remarks on the Marcinkiewicz–Orlicz space*, Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 1–3.

[Al59] J. Albrycht, *Some remarks on the Marcinkiewicz–Orlicz space. II, III*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 11–12 and 55–56.

[Al62] J. Albrycht, *The theory of Marcinkiewicz–Orlicz spaces*, Dissertationes Math. Rozprawy Mat. 27 (1962), 1–56.

[AG05] F. Andreano, R. Grande, *The Hilbert transform on Marcinkiewicz spaces*, Afr. J. Math. Phys. 2 (2005), no. 2, 111–118.

[ABG06] J. Andres, A. M. Bersani, R. F. Grande, *Hierarchy of almost-periodic function spaces*, Rend. Mat. Appl. (7) 26 (2006), 121–188 [Marcinkiewicz spaces, pp. 153–154].

[Ba63] J. Bass, *Espaces de Besicovitch, fonctions presque-périodiques, fonctions pseudo-aléatoires*, Bull. Soc. Math. France 91 (1963), 39–61 [Marcinkiewicz theorem on completeness of M^2, p. 40].

[Be65] V. E. Beneš, *A nonlinear integral equation in the Marcinkiewicz space* $\mathfrak{m}_2$, J. Math. and Phys. 44 (1965), 24–35.

[Be66] J.-P. Bertrandias, *Espaces de fonctions bornées et continues en moyenne asymptotique d'ordre p*, Bull. Soc. Math. France Mém. 5 (1966) [Besicovitch–Marcinkiewicz space M^p, pp. 12–51; Marcinkiewicz theorem on completeness, p. 12].

[BF45] H. Bohr, E. Følner, *On some types of functional spaces. A contribution to the theory of almost periodic functions*, Acta Math. 76 (1945), 31–155.

[CL06] G. Cohen, V. Losert, *On Hartman almost periodic functions*, Studia Math. 173 (2006), 81–101. 83–85].

[HW47] P. Hartman, A. Wintner, *The* (L_2)*-space of relative measure*, Proc. Nat. Acad. Sci. U.S.A. 33 (1947), 128–132.

[He99] C. Heil, *The Wiener transform on the Besicovitch spaces*, Proc. Amer. Math. Soc. 127 (1999), 2065–2071.

[Hi86] T. R. Hillmann, *Besicovitch–Orlicz spaces of almost periodic functions*, in: Real and Stochastic Analysis, Wiley, New York, 1986, 119–167.

[KP95] O. V. Kucher, A. M. Plichko, *Limits on the real line of symmetric spaces on segments*, Ukraïn. Mat. Zh. 47 (1995), 46–55; English transl.: Ukrainian Math. J. 47 (1995), 50–62.

[KPl95] O. V. Kucher, A. M. Plichko, *The Wiener transformation on the limits of symmetric spaces*, Acta Univ. Carolin. Math. Phys. 36 (1995), no. 2, 39–52.

[La80] K.-S. Lau, *On the Banach spaces of functions with bounded upper means*, Pacific J. Math. 91 (1980), 153–172.

[La81] K.-S. Lau, *The class of convolution operators on the Marcinkiewicz spaces*, Ann. Inst. Fourier (Grenoble) 31 (1981), 225–243.

[La83] K.-S. Lau, *Extension of Wiener's Tauberian identity and multipliers on the Marcinkiewicz space*, Trans. Amer. Math. Soc. 277 (1983), 489–506.

[LZ63] W. A. J. Luxemburg, A. C. Zaanen, *Notes on Banach function spaces. I*, Indag. Math. 25 (1963), 135–147.

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[Sa71] U. Santarelli, *Un modello astratto dello spazio di Besicovič–Marcinkiewicz*, Boll. Un. Mat. Ital. (4) 4 (1971), 907–911.

[Ur61] K. Urbanik, *Fourier analysis in Marcinkiewicz spaces*, Studia Math. 21 (1961), 93–102.

[VT77] M. Vidyasagar, M. A. L. Thathachar, *A note on feedback stability and instability in the Marcinkiewicz space* M_2, IEEE Trans. Circuits and Systems CAS-24 (1977), no. 3, 127–131.

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4.1.4. *Marcinkiewicz–Zygmund vector-valued inequalities* (1939). Marcinkiewicz–Zygmund vector-valued inequalities are estimates of operators between vector-valued spaces.

THEOREM 3 (Marcinkiewicz–Zygmund 1939). *For an arbitrary linear bounded operator* $T : L^p \to L^p$ *between real or complex (quasi-)normed Lebesgue spaces we have vector-*

valued estimate with constant 1*, that is,*

$$\Big\|\Big(\sum_{k=1}^{n}|Tf_k|^2\Big)^{1/2}\Big\|_p \le \|T\|_{L^p\to L^p}\Big\|\Big(\sum_{k=1}^{n}|f_k|^2\Big)^{1/2}\Big\|_p \tag{12}$$

and for $0<p<r\le 2$

$$\Big\|\Big(\sum_{k=1}^{n}|Tf_k|^r\Big)^{1/r}\Big\|_p \le \|T\|_{L^p\to L^p}\Big\|\Big(\sum_{k=1}^{n}|f_k|^r\Big)^{1/r}\Big\|_p \tag{13}$$

for arbitrary $f_1,f_2,\dots,f_n\in L^p(\mu)$ *and any* $n\in\mathbb{N}$*.*

The estimates (12) and (13) were proved by Marcinkiewicz–Zygmund with the help of Gaussian variables and r-stable Gaussian variables in paper [MZ39a, Thms 1 and 2]. These theorems together with their proof appeared in books by Edwards–Gaudry [EG77, pp. 203–204], Grafakos [Gr08, pp. 316–318] and in a paper by Andersen [An80]. The inequality (12) with the proof is also in the book by Nikolski [Ni02] and without proof in the book by Beckenbach and Bellman [BB83, p. 39].

More general, for an arbitrary linear bounded operator $T: L^p\to L^p$ between real or complex (quasi-)normed Lebesgue spaces with arbitrary σ-finite measures μ and ν, and for $0<p,q,r\le\infty$ and natural $n\ge 2$ let $K_{p,q}^{(n)}(r)$ be the smallest constant $C\ge 1$ in the inequality

$$\Big\|\Big(\sum_{k=1}^{n}|Tf_k|^r\Big)^{1/r}\Big\|_q \le C\|T\|_{L^p\to L^q}\Big\|\Big(\sum_{k=1}^{n}|f_k|^r\Big)^{1/r}\Big\|_p \tag{14}$$

for arbitrary $f_1,f_2,\dots,f_n\in L^p(\mu)$.

The properties of constants $K_{p,q}^{(n)}(r)$ and $K_{p,q}(r)=\sup_{n\ge 2}K_{p,q}^{(n)}(r)$ for $1\le p,q,r\le\infty$ were investigated by Marcinkiewicz and Zygmund (1939), Herz (1971), Krivine (1978, 1979), Defant and Floret (1993), Gasch and Maligranda (1994), Vogt (1995), Defant and Junge (1997), Maligranda and Sabourova (2011).

The equalities $K_{p,p}(2)=1$ and $K_{p,p}(r)=1$, where $0<p<r\le 2$, are just results of Marcinkiewicz–Zygmund. Note that $K_{p,q}^{(n)}(r)$ are increasing in n and p, but decreasing in q. Moreover, if $0<p\le q\le\infty$, then $K_{p,q}(2)=1$.

Using the equivalence (10) we can easily prove that if $0<p,q<\infty$ and $T: L^p\to L^{q,\infty}$ is a bounded linear operator, then

$$\Big\|\Big(\sum_{k=1}^{n}|Tf_k|^2\Big)^{1/2}\Big\|_{q,\infty} \le C\|T\|_{L^p\to L^{q,\infty}}\Big\|\Big(\sum_{k=1}^{n}|f_k|^2\Big)^{1/2}\Big\|_p$$

for arbitrary $f_1,f_2,\dots,f_n\in L^p(\mu)$ and any $n\in\mathbb{N}$. Moreover, $C\le K_{p,r}^{(n)}(2)(\frac{q}{q-r})^{1/r}$ for any $0<r<q$.

Constants $K_{p,q}(2)$ are connected with the relation between norm of the operator $T: L^p_{\mathbb{R}}\to L^q_{\mathbb{R}}$ and the norm of its natural complexification between complex spaces $T_{\mathbb{C}}: L^p_{\mathbb{C}}\to L^q_{\mathbb{C}}$ given by the formula $T_{\mathbb{C}}(f+ig)=T(f)+i\,T(g)$ (cf. [Ve76], [Kr77], [DF93], [Vo95] and [MS11]).

Among books on this topic the following ones should be mentioned:

[BB83] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer, Berlin, 1983 [Theorem 12 of Zygmund and Marcinkiewicz, p. 30].

[DF93] A. Defant, K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993 [Marcinkiewicz–Zygmund result, pp. 314, 315, 347].

[EG77] R. E. Edwards, G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Ergeb. Math. Grenzgeb. 90, Springer, Berlin, 1977 [Marcinkiewicz–Zygmund theorem, pp. 203–204].

[GR85] J. García-Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies 116, North-Holland, Amsterdam, 1985 [V.2. A theorem of Marcinkiewicz and Zygmund, pp. 482–487].

[Ni02] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading, Vol. 1. Hardy, Hankel, and Toeplitz*, Amer. Math. Soc., Providence, 2002 [(k) Marcinkiewicz and Zygmund (1939), p. 120].

[Gr08] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Grad. Texts Math. 249, Springer, New York, 2008 [4.5.1. Marcinkiewicz–Zygmund theorem, pp. 316–319].

Papers discussing vector-valued Marcinkiewicz–Zygmund inequalities are:

[An80] K. F. Andersen, *Inequalities for scalar-valued linear operators that extend to their vector-valued analogues*, J. Math. Anal. Appl. 77 (1980), 264–269.

[CS83] M. Cotlar, C. Sadosky, *Vector valued inequalities of Marcinkiewicz–Zygmund and Grothendieck type for Toeplitz forms*, in: Harmonic Analysis (Cortona, 1982), Lecture Notes in Math. 992, Springer, Berlin, 1983, 276–308.

[DJ97] A. Defant, M. Junge, *Best constants and asymptotics of Marcinkiewicz–Zygmund inequalities*, Studia Math. 125 (1997), 271–287.

[GM94] J. Gasch, L. Maligranda, *On vector-valued inequalities of the Marcinkiewicz–Zygmund, Herz and Krivine type*, Math. Nachr. 167 (1994), 95–129.

[He71] C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. 154 (1971), 69–82.

[Kr77] J.-L. Krivine, *Sur la complexification des opérateurs de L^∞ dans L^1*, C. R. Acad. Sci. Paris Sér. A–B 284 (1977), A377–A379.

[Kr78] J.-L. Krivine, *Constantes de Grothendieck et fonctions de type positif sur les sphères*, in: Séminaire sur la géométrie des espaces de Banach (1977–1978), École Polytech., Palaiseau, 1978, Exp. No. 1–2.

[Kr79] J.-L. Krivine, *Constantes de Grothendieck et fonctions de type positif sur les sphères*, Adv. in Mat. 31 (1979), 16–30.

[MS11] L. Maligranda, N. Sabourova, *Real and complex operator norms between quasi-Banach $L^p - L^q$ spaces*, Math. Inequal. Appl. 14 (2011), 247–270.

[Pi11] G. Pisier, *Grothendieck's Theorem, past and present*, http://arxiv.org/abs/1101.4195 [Marcinkiewicz–Zygmund inequality, p. 9].

[RS91] H. P. Rosenthal, S. J. Szarek, *On tensor products of operators from L^p to L^q*, in: Functional Analysis (Austin, TX, 1987/1989), Lecture Notes in Math. 1470, Springer, Berlin, 1991, 108–132.

[Ve76] I. E. Verbickiĭ, *Some relations between the norm of an operator and that of its complex extension*, Mat. Issled. 42 (1976), 3–12 (Russian).

[Vo95] H. Vogt, *Komplexifizierung von Operatoren zwischen L_p-Räumen*, Diplomarbeit, Carl von Ossietzky Universität Oldenburg, Oldenburg, 1995.

4.1.5. *Rearrangements of series – Marcinkiewicz example* (1936). Let X be a Banach space and $\sum_{k=1}^{\infty} x_k$ be a series in X. Denote by $S = S(\sum_{k=1}^{\infty} x_k)$ the *set of sums* of this series, that is, the set $\{x \in X$: there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $x = \sum_{k=1}^{\infty} x_{\pi(k)}\}$.

If $X = \mathbb{R}$, then S is either empty (divergent series) or single point (absolutely convergent series) or whole $\mathbb{R}$ (for any conditionally convergent series by the Riemann theorem). If $X = \mathbb{C}$, then we have four alternatives on $S(\sum_{k=1}^{\infty} x_k)$: empty set, single point (absolutely convergent series), straight line in $\mathbb{C}$ (for example, $S = \mathbb{R} + ia$ for $\sum_{k=1}^{\infty}[\frac{(-1)^{k+1}}{k} + \frac{ia}{k(k+1)}]$, $a \in \mathbb{R}$ fixed) and the whole $\mathbb{C}$ (if $x_{2k} \in \mathbb{R}$, $x_{2k-1} \in \mathbb{R}$ for any k and each of series $\sum x_{2k}$, $\sum x_{2k-1}$ is a conditionally convergent series).

For finite-dimensional X the famous Lévy–Steinitz theorem on rearrangements of series gives that $S(\sum_{k=1}^{\infty} x_k)$ is a linear manifold in X, that is, $S = x_0 + M$, where $x_0 \in X$ and M is a linear subspace of X. The theorem was first proven by P. Lévy in 1905. In 1913 E. Steinitz pointed out that Lévy proof was incomplete, especially in the higher-dimensional cases. Steinitz filled the gap of Lévy's proof and also found an entirely different approach (cf. [Ro87]).

Already in 1927, the paper written by Orlicz [Or27] contained on page 124 the Banach question on convexity of the set of sums in infinite-dimensional spaces. In 1935, in Problem 106 of the "Scottish Book", Banach asked whether Lévy–Steinitz theorem is valid in infinite-dimensional normed spaces. Banach proposed to prove that for any series in a Banach space its set of sums is a linear manifold. A simple and elegant counter-example in $L^2[0,1]$ to this conjecture was given by J. Marcinkiewicz. The solution by Marcinkiewicz also appears in the Scottish Book and the answer was negative. Marcinkiewicz constructed an example of a conditionally convergent series in infinite-dimensional Hilbert space $L^2[0,1]$ with even a nonconvex set of sums since series of integer-valued functions cannot converge in the strong L^2 topology to $1/2$.

Theorem 4 (Marcinkiewicz 1936). *The Lévy–Steinitz theorem does not hold in $L^2[0,1]$ space since there exists a series in $L^2[0,1]$ such that the set of sums S is a nonconvex set.*

As the proof we present Marcinkiewicz's construction ([Mau81], p. 188): in $L^2[0,1]$ consider a sequence

$$x_{2^n+k} = \chi_{[k/2^n,(k+1)/2^n]}, \text{ where } 0 \le n < \infty,\ 0 \le k < 2^n.$$

Then $x_1 = \chi_{[0,1]} = 1$, $x_2 = \chi_{[0,1/2]}$, $x_3 = \chi_{[1/2,1]}$, $x_4 = \chi_{[0,1/4]}$, $x_5 = \chi_{[1/4,1/2]}$, $x_6 = \chi_{[1/2,3/4]}$, $x_7 = \chi_{[3/4,1]}$, $x_8 = \chi_{[0,1'8]}$, etc. Consider the series $\sum_{n=1}^{\infty} y_n$, where $y_{2n-1} = x_n$ and $y_{2n} = -x_n$ $(n \ge 1)$. Since $\|x_{2^n+k}\|_2^2 = 2^{-n} \to 0$ as $n \to \infty$ it follows that $\sum_{n=1}^{\infty} y_n = (x_1 - x_1) + (x_2 - x_2) + \ldots = 0$. Also since $x_2 + x_3 - x_1 = x_4 + x_5 - x_2 = x_6 + x_7 - x_3 = \ldots = 0$ it follows that $x_1 + (x_2 + x_3 - x_1) + (x_4 + x_5 - x_2) + \ldots = 1$. However, no rearrangement will make that the series converge to the function identically equal to $\frac{1}{2}$ on $[0,1]$, because each of the partial sums of the series is an integer-valued function. Thus, the set of sums S is not a convex set since $0, 1 \in S$ but $\frac{1}{2} \notin S$. Of course, any constant function l, $0 < l < 1$ is not in S. ∎

Note that Marcinkiewicz's construction will show nonconvexity of the set of sums S also in $L^p[0,1]$ for $0 < p < \infty$, in $C[0,1]$ which was mentioned by Marcinkiewicz (since by the Banach theorem [Ba32, p. 185] the space $L^2[0,1]$ can be imbedded isometrically in the Banach space $C[0,1]$) and in $L^\infty[0,1]$ (since it has $C[0,1]$ as a subspace).

Examples of series in $L^p[0,1]$ with nonconvex set of sums S were given independently in 1971 by Nikishin [Ni71] (for $p = \infty$) and in 1980 by Kornilov [Ko80] (for $1 \le p \le 2$).

Kadets [Ka86], making use of the Marcinkiewicz–Kornilov example together with the Dvoretzky theorem, has shown that in any infinite-dimensional Banach space there exists a series with nonlinear set of sums S (more precisely, that the set of sums of convergent rearrangements of the series fails to be convex). Today we know more: for any fixed elements x, y of an infinite-dimensional Banach space there exists a conditionally convergent series $\sum_{k=1}^{\infty} x_k$ such that $S(\sum_{k=1}^{\infty} x_k) = \{x, y\}$ (cf. [KW89]). Moreover, for a given finite subset A of an infinite-dimensional Banach space X there is a series in X whose sum range equals A (cf. [Wo05]).

Among books, papers and bibliographies discussing the Lévy–Steinitz theorem, the Banach problem and the Marcinkiewicz example are:

[Mau81] R. D. Mauldin, *The Scottish Book. Mathematics from the Scottish Café*, Birkhäuser, Boston, 1981 [Problem 106 of Banach, Marcinkiewicz example and commentary by R. D. Mauldin and W. A. Beyer, pp. 188–190].

[KK91] V. M. Kadets, M. I. Kadets, *Rearrangements of Series in Banach Spaces*, Transl. Math. Monogr. 86, Amer. Math. Soc., Providence, 1991 [Marcinkiewicz example, pp. 24–25].

[DJT95] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press, Cambridge, 1995 [Marcinkiewicz example, p. 21].

[KK97] M. I. Kadets, V. M. Kadets, *Series in Banach Spaces. Conditional and Unconditional Convergence*, Oper. Theory Adv. Appl. 94, Birkhäuser, Basel, 1997 [Marcinkiewicz's construction, pp. 30–31].

[Ch05] A. G. M. Champi, *Absolute and unconditional convergence in Banach spaces*, Master Thesis, Rio de Janeiro, December 2005 (Portuguese) [4.4. Marcinkiewicz counter-example, pp. 54–56].

[CC97] M.-J. Chasco, S. Chobanyan, *On rearrangements of series in locally convex spaces*, Michigan Math. J. 44 (1997), 607–617. 608].

[CG89] S. Chobanyan, G. J. Georgobiani, *A problem on rearrangements of summands in normed spaces and Rademacher sums*, in: Probability Theory on Vector Spaces, IV (Łańcut, 1987), Lecture Notes in Math. 1391, Springer, Berlin, 1989, 33–46 [information on Marcinkiewicz counter-example, p. 42].

[Ha86] I. Halperin, *Sums of a series, permitting rearrangements*, C. R. Math. Rep. Acad. Sci. Canada 8 (1986), 87–102 [9. The counter-example of Marcinkiewicz, p. 100].

[HA89] I. Halperin, T. Ando, *Bibliography: series of vectors and Riemann sums*, Hokkaido Univ., Research Inst. of Applied Electricity, Division of Applied Mathematics, Sapporo, 1989 [information on counter-example of J. Marcinkiewicz from 1937 in the preface].

[Ka86] V. M. Kadets, *A problem of S. Banach* (*problem* 106 *from the "Scottish Book"*), Funktsional. Anal. i Prilozhen. 20 (1986), no. 4, 74–75; English transl.: Functional Anal. Appl. 20 (1986), 317–319.

[Ka89] V. M. Kadets, *Series permutation in infinite-dimensional spaces* (*main results and open problems*), C. R. Math. Rep. Acad. Sci. Canada 11 (1989), no. 5, 151–164 [information on counter-example in $L_2[0,1]$ by J. Marcinkiewicz, p. 155].

[KW89] M. I. Kadets, K. Woźniakowski, *On series whose permutations have only two sums*, Bull. Polish Acad. Sci. Math. 37 (1989), 15–21 [information that Marcinkiewicz has given a simple counter-example, p. 15].

[Ko80] P. A. Kornilov, *On rearrangements of conditionally convergent functional series*, Mat. Sb. (N.S.) 113 (1980), 598–616 (Russian).

[N71] E. M. Nikishin, *A certain problem of Banach*, Dokl. Akad. Nauk SSSR 196 (1971), 774–775; English transl.: Soviet Math. Dokl. 12 (1971), 255–257.

[Ni71] E. M. Nikishin, *Rearrangements of function series*, Mat. Sb. (N.S.) 85 (1971), 272–285; English transl.: Math. USSR-Sb. 14 (1971), 267–280.

[Or27] W. Orlicz, *Über die unabhängig von der Anordnung fast überall konvergenten Funktionenreihen*, Bull. Internat. Acad. Polon. Sci. Sér. A 1927, 117–125; reprinted in: Władysław Orlicz, Collected Papers I, 41–49 [Banach problem on convexity of S on page 124].

[Ro87] P. Rosenthal, *The remarkable theorem of Lévy and Steinitz*, Amer. Math. Monthly 94 (1987), 342–351. 350].

[So08] M. A. Sofi, *Levy–Steinitz theorem in infinite dimension*, New Zealand J. Math. 38 (2008), 63–73 [information on Marcinkiewicz example in $L_2[0,1]$, p. 63].

[Wo05] J. O. Wojtaszczyk, *A series whose sum range is an arbitrary finite set*, Studia Math. 171 (2005), 261–281. [information on Marcinkiewicz example in $L_2[0,1]$, p. 261].

4.2. Probability theory. In the years 1937–1938 Marcinkiewicz was interested in independent random variables. He called them independent functions. Papers, joint with Zygmund [MZ37c], [MZ38a], and his own papers [M38b], [M38c] and [M38e] are discussing problems about these functions.

4.2.1. *Marcinkiewicz–Zygmund inequalities for independent random variables* (1937). Consider the Rademacher functions $r_k(t) = \operatorname{sign}[\sin(2^k\pi t)]$, $k \in \mathbb{N}$, $t \in [0,1]$, which form an orthonormal system in $L^2[0,1]$, that is, we have

$$\int_0^1 r_k(t)\,dt = 0, \quad \int_0^1 r_k(t)^2\,dt = 1 \quad \text{and} \quad \int_0^1 r_k(t)\,r_m(t)\,dt = 0 \text{ for } k \neq m.$$

Immediately from here we get equalities

$$\begin{aligned}\int_0^1 \Big|\sum_{k=1}^n r_k(t)a_k\Big|^2\,dt &= \int_0^1 \Big[\sum_{k=1}^n r_k(t)a_k\Big]\,\Big[\sum_{m=1}^n r_m(t)a_m\Big]\,dt\\ &= \sum_{k=1}^n\sum_{m=1}^n a_k a_m \int_0^1 r_k(t)r_m(t)\,dt = \sum_{k=1}^n\sum_{m=1}^n a_k a_m \delta_{km} = \sum_{k=1}^n a_k^2,\end{aligned}$$

and for L^p space the Khintchine inequality (inequalities) from 1923 reads: *for $p \in \mathbb{R}$, $p > 0$ there exist constants $A_p, B_p > 0$ such that*

$$A_p\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \le \Big(\int_0^1 \Big|\sum_{k=1}^n r_k(t)a_k\Big|^p\,dt\Big)^{1/p} \le B_p\Big(\sum_{k=1}^n |a_k|^2\Big)^{1/2} \tag{15}$$

for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and any $n \in \mathbb{N}$.

Rademacher functions are also independent random variables on $[0,1]$. More general, consider random variables, that is, measurable functions $X_k : \Omega \to \mathbb{R}$ on the probability space (Ω, Σ, P) and assume that they are independent, i.e., for any intervals $I_1, I_2, \ldots, I_n \subset \mathbb{R}$ we have the equality

$$P\big(\{t \in \Omega : X_1(t) \in I_1, X_2(t) \in I_2, \ldots, X_n(T) \in I_n\}\big) = \prod_{k=1}^n P\big(\{t \in \Omega : X_k(t) \in I_k\}\big).$$

Khintchine inequalities were generalized by Marcinkiewicz and Zygmund in 1937 in their paper [MZ37e] (see also [KS84], Theorems 2.5 and 2.6, and [AS10], Theorems 2 and 3) to uniformly bounded random variables on $[0,1]$.

THEOREM 5 (Marcinkiewicz–Zygmund inequalities 1937). *Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables on $[0,1]$ satisfying the conditions*

$$\int_0^1 X_k(t)\,dt = 0, \quad \|X_k\|_2 = \Big(\int_0^1 X_k(t)^2\,dt\Big)^{1/2} = 1, \quad \|X_k\|_\infty \le M \quad (k \in \mathbb{N}). \tag{16}$$

Then, for any $a = (a_k)_{k=1}^n \in \mathbb{R}$, any $n \in \mathbb{N}$ and $1 \le p < \infty$, we have:

$$m\Big\{t \in [0,1] : \Big|\sum_{k=1}^n a_k\, X_k(t)\Big| > \lambda\|a\|_2\Big\} \le 2\exp(-\lambda^2/(4M^2)) \quad \textit{for any } \lambda > 0, \tag{17}$$

$$\frac{1}{C}\Big(\sum_{k=1}^n a_k^2\Big)^{1/2} \le \Big\|\sum_{k=1}^n a_k X_k\|_p \le C\Big(\sum_{k=1}^n a_k^2\Big)^{1/2} \quad \textit{for some } C = C_{M,p} > 0, \tag{18}$$

and

$$\int_0^1 \exp\Big(\lambda \max_{1\le m\le n}\Big|\sum_{k=1}^m a_k X_k(t)\Big|\Big)\,dt \le 32\exp\Big(\frac{1}{2}\lambda^2\sum_{k=1}^n a_k^2\Big) \quad \textit{for any } \lambda > 0. \tag{19}$$

We can find the proof of the inequalities (17) and (18) in the book [KS84, pp. 39–41]. Inequality (17) is also called the *Hoeffding inequality* (especially by probabilists and statisticians) however it was published only in 1963 by Wassily Hoeffding in [Ho63]. Note that already in 1929 estimates similar to (17) were proved by Kolmogorov [Ko29, p. 127]. A proof of estimate (19) can be found in [MZ37e], [Ka72, pp. 571–573] and [Ts51, p. 143 for Rademacher functions].

In 2000 Astashkin [As00] proved that the system of independent random variables satisfying (16) is even equivalent in the distribution sense to the Rademacher system (see also [As09], Theorem 8.4 and Corollary 8.3).

Another generalization of the Khintchine inequality for the sum of random variables was given by Marcinkiewicz and Zygmund for $p > 1$ in the paper [MZ37c, Thm 13, p. 87] from 1937 and for $p \ge 1$ in the paper [MZ38a, Theorem 5, p. 109] from 1938. Falsity of the inequalities for $0 < p < 1$ was also shown by Marcinkiewicz and Zygmund in [MZ38a, pp. 112–113].

THEOREM 6 (Marcinkiewicz–Zygmund inequalities 1937). *Let $1 \le p < \infty$. Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables with $E(X_k) = \int_\Omega X_k(t)\,dP = 0$ and such that $E(|X_k|^p) = \int_\Omega |X_k(t)|^p\,dP < \infty$ for any $k \in \mathbb{N}$. Then there are constants $A_p^*, B_p^* > 0$ such that for any $n \in \mathbb{N}$ we have the inequalities*

$$A_p^*\Big\|\Big(\sum_{k=1}^n X_k^2\Big)^{1/2}\Big\|_p \le \Big\|\sum_{k=1}^n X_k\Big\|_p \le B_p^*\Big\|\Big(\sum_{k=1}^n X_k^2\Big)^{1/2}\Big\|_p. \tag{20}$$

For $0 < p < 1$ no one of these two inequalities (20) is true.

Burkholder (1988) proved that for $1 < p < \infty$ we have $B_p^* \le \max(p-1, \frac{1}{p-1})$ and the constant $B_p^* = p - 1$ is sharp for $p \ge 2$.

From the second inequality in (20) and from the Hölder–Rogers inequality we are getting that for $p \geq 2$

$$\Big\|\sum_{k=1}^{n} X_k\Big\|_p \leq C_p\, n^{1/2-1/p}\Big(\sum_{k=1}^{n}\|X_k\|_p^p\Big)^{1/p} \tag{21}$$

or, equivalently, for $p \geq 1$

$$E\Big(\Big|\sum_{k=1}^{n} X_k\Big|^{2p}\Big) \leq C_{2p}^{2p}\, n^{p-1}\sum_{k=1}^{n} E(|X_k|^{2p}). \tag{21$'$}$$

Inequality (21) or (21′) is sometimes also called *Marcinkiewicz–Zygmund inequality*, about which it was written by Petrov [Pe95, Theorem 2.10, p. 62] (see also [CMR05, p. 292]), and Ren–Liang [RL01, pp. 228], where they proved that $C_p \leq 3\sqrt{2p}$. Combining the result of Burkholder and Ren–Liang we receive that $C_p \leq \min(B_p^*, 3\sqrt{2p}\,)$.

Burkholder and Gundy (1970) generalized the Marcinkiewicz–Zygmund inequality for the modular inequalities. Let $\Phi : [0,\infty) \to [0,\infty)$ be a convex function with $\Phi(0) = 0$, satisfying the condition Δ_2, that is, $\Phi(2u) \leq C\Phi(u)$ for any $u > 0$. If $(X_k)_{k=1}^{\infty}$ is a sequence of independent random variables such that $E(X_k) = \int_\Omega X_k(t)\,dP = 0$, then there are constants $A, B > 0$ dependent only on Φ such that for any $n \in \mathbb{N}$ the following inequalities hold:

$$A\int_\Omega \Phi\Big[\Big(\sum_{k=1}^{n} X_k^2\Big)^{1/2}\Big]\,dP \leq \int_\Omega \Phi\Big(\Big|\sum_{k=1}^{n} X_k\Big|\Big)\,dP \leq B\int_\Omega \Phi\Big[\Big(\sum_{k=1}^{n} X_k^2\Big)^{1/2}\Big]\,dP. \tag{20$'$}$$

Ingenious proofs of generalized Khintchine inequality (15) for integral modular can be found in [Ka77, Lemma 6.2] and [Ma89, Sublemma 14.6(b)]: there are constants $C, D > 0$ dependent on Φ such that the following inequalities are true:

$$C\Phi\Big[\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}\Big] \leq \int_0^1 \Phi\Big(\Big|\sum_{k=1}^{n} a_k r_k(t)\Big|\Big)\,dt \leq D\Phi\Big[\Big(\sum_{k=1}^{n} a_k^2\Big)^{1/2}\Big] \tag{15$'$}$$

for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and arbitrary $n \in \mathbb{N}$. Of course, the inequalities (15′) are special case of inequalities (20′).

Johnson and Schechtman (1988) proved a generalization of Marcinkiewicz–Zygmund inequalities (20) on symmetric spaces. If X is a symmetric space on $[0,1]$, which either is separable or has the Fatou property and the lower Boyd index $\alpha_X > 0$, then the inequality of the Marcinkiewicz–Zygmund type

$$\Big\|\sum_{k=1}^{n} X_k\Big\|_X \leq C\Big\|\Big(\sum_{k=1}^{n} X_k^2\Big)^{1/2}\Big\|_X \quad (n = 1, 2, \ldots) \tag{22}$$

holds for any sequence of independent random variables $(X_k)_{k=1}^{\infty} \subset X$ with $\int_0^1 X_k(t)\,dt = 0$ $(k = 1, 2, \ldots)$. In fact, Johnson and Schechtman (1988) proved that inequality (22) holds for any sequence of martingale differences $(X_k)_{k=1}^{\infty} \subset X$ if and only if the lower Boyd index $\alpha_X > 0$.

Astashkin (2008) showed that instead of assumption $\alpha_X > 0$ the necessary and sufficient condition for inequality (22) to hold is the Kruglov property of the space X, introduced by M. Sh. Braverman by using some probabilistic construction of V. M. Kruglov

(1970). This property is satisfied by spaces which are sufficiently "far" from L^∞ – the smallest symmetric space on $[0,1]$. For example, the symmetric space X has the Kruglov property if $X \supset L^p$ for some $p < \infty$ (in particular, if $\alpha_X > 0$). A more interesting example is the exponential Orlicz spaces L^M, generated by the functions $M(u) = \exp(u^p) - 1$ for $1 \le p < \infty$, which are "near" to the space L^∞ and do not contain L^q for any $q < \infty$ (Braverman 1994).

Another powerful inequality was proved by Rosenthal (1970): *Let* $2 \le p < \infty$ *and let* $(X_k)_{k\in\mathbb{N}}$ *be independent random variables such that* $E(X_k) = 0$ *and* $E(|X_k|^p) < \infty$ *for any* $k \in \mathbb{N}$. *Then there exist constants* $C_p, D_p > 0$ *such that for any* $n \in \mathbb{N}$ *the following inequalities hold*:

$$C_p \max\bigg(\Big(\sum_{k=1}^n \|X_k\|_{L^p}^p\Big)^{1/p}, \Big(\sum_{k=1}^n \|X_k\|_{L^2}^2\Big)^{1/2}\bigg) \le \Big\|\sum_{k=1}^n X_k\Big\|_{L^p}$$
$$\le D_p \max\bigg(\Big(\sum_{k=1}^n \|X_k\|_{L^p}^p\Big)^{1/p}, \Big(\sum_{k=1}^n \|X_k\|_{L^2}^2\Big)^{1/2}\bigg).$$

The Marcinkiewicz–Zygmund inequalities (20) can be found, e.g., in the books by Kawata ([Ka72], Theorem 13.6.1) and Gut ([Gu05], Theorem 8.1). Gut, in fact, gave the proof of the Marcinkiewicz–Zygmund inequalities with the help of Khintchine inequality with constants $A_p^* = A_p^{1/p}/2$, $B_p^* = 2B_p^{1/p}$ (pp. 150–151), and the Rosenthal inequality is proved with the help of Marcinkiewicz–Zygmund inequalities (pp. 151–153).

Marcinkiewicz and Zygmund proved also other inequalities in the papers [MZ37c] and [MZ38a, Theorem 5]. For example, in the first paper in Theorems 1 and 3 we have the following estimates:

Theorem 7 (Marcinkiewicz–Zygmund inequalities 1937). *Let* X_k *be independent random variables such that* $E(X_k) = 0$ *for* $k = 1, 2, \dots, n$ *and* $S_n = \sum_{k=1}^n X_k$.

(a) *If* $p > 1$, *then*

$$\Big\|\max_{1\le m\le n} |S_m|\,\Big\|_p \le 2^{1/p}\,\frac{p}{p-1}\,\|S_n\|_p. \tag{23}$$

(b) *If* $EX_k^2 = 1$ *and* $E|X_k| \ge \alpha > 0$ *for* $k = 1, 2, \dots, n$, *then there exists a constant* $C = C(\alpha) > 0$ *such that for any* $(a_k)_{k=1}^n$ *we have*

$$\Big\|\max_{1\le m\le n}\Big|\sum_{k=1}^m a_k X_k\Big|\Big\|_1 \le C(\alpha)\Big\|\sum_{k=1}^n a_k X_k\Big\|_1. \tag{24}$$

In the proof of Theorem 7(a) they are using the Hardy–Littlewood result (1930) on the maximal function. In the proof of part (b) they use the result from (a) for $p = 2$ and the Paley–Zygmund inequality ([PZ32], p. 192). Moreover, Mogyoródi [Mo79] generalized the inequality (23) to the form $\big\|\max_{1\le m\le n}\Phi(|S_m|)\big\|_1 \le C\|\Phi(|S_n|)\|_1$, where Φ is a convex Young function satisfying together with its complementary function the so-called Δ_2-condition.

Let us collect some books and papers quoted above or devoted to this subject:

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Academic Press, New York–London, 1972 [13.5.3. Marcinkiewicz–Zygmund theorem, pp. 568–571; 13.6.1. Marcinkiewicz–Zygmund, pp. 576–578].

[St74] W. F. Stout, *Almost Sure Convergence*, Academic Press, New York–London, 1974 [Theorem 3.3.6. Martingale version of Marcinkiewicz and Zygmund inequality, pp. 149–152].

[Pe75] V. V. Petrov, *Sums of Independent Random Variables*, Ergeb. Math. Grenzgeb. 82, Springer, New York, 1975 [pp. 59–60].

[KS84] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (Russian) [Marcinkiewicz–Zygmund inequalities (18), (17): Theorems 2.5 and 2.6, pp. 39–42].

[SW86] G. R. Shorack, J. A. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, New York, 1986 [Appendix 5. Marcinkiewicz and Zygmund equivalences, pp. 858–859].

[CT88] Y. S. Chow, H. Teicher, *Probability Theory. Independence, Interchangeability, Martingales*, 2nd ed., Springer, New York, 1988 [10.3 Marcinkiewicz–Zygmund inequality, pp. 366–369; 11.2. Martingale extension of Marcinkiewicz–Zygmund inequalities, pp. 394–402].

[Pe95] V. V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford Stud. Probab. 4, Oxford Univ. Press, New York, 1995 [2.6.18. Marcinkiewicz–Zygmund inequalities, pp. 82; Theorem 2.10 is a corollary from Marcinkiewicz–Zygmund inequalities, pp. 62 and 77].

[DG99] V. H. de la Peña, E. Giné, *Decoupling. From Dependence to Independence. Randomly Stopped Processes. U-statistics and Processes. Martingales and Beyond*, Springer, New York, 1999 [Lemma 1.4.13. Marcinkiewicz inequalities, pp. 34–35].

[CMR05] O. Cappé, E. Moulines, T. Rydén, *Inference in Hidden Markov Models*, Springer, New York, 2005 [9.1.5. Marcinkiewicz–Zygmund inequality, p. 292].

[Gu05] A. Gut, *Probability: A Graduate Course*, Springer Texts Stat., Springer, New York, 2005 [8.1. Marcinkiewicz–Zygmund inequalities, pp. 150–151].

[As00] S. V. Astashkin, *Systems of random variables equivalent in distribution to the Rademacher systems, and the $\mathcal{K}$-closed representability of Banach pairs*, Mat. Sb. 191 (2000), no. 6, 3–30; English transl.: Sb. Math. 191 (2000), 779–807.

[As08] S. V. Astashkin, *Independent functions in symmetric spaces and the Kruglov property*, Mat. Sb. 199 (2008), no. 7, 3–20; English transl.: Sb. Math. 199 (2008), 945–963.

[As09] S. V. Astashkin, *Rademacher functions in symmetric spaces*, Sovrem. Mat. Fundam. Napravl. 32 (2009), 3–161; English transl.: J. Math. Sci. (N.Y.) 169 (2010), 725–886.

[AS10] S. V. Astashkin, F. A. Sukochev, *Independent functions and geometry of Banach spaces*, Uspekhi Mat. Nauk 65 (2010), no. 6, 3–86; English transl.: Russian Math. Surveys 65 (2010), 1003–1081 [Marcinkiewicz–Zygmund inequalities (18), (17): Theorems 2 and 3].

[BBLM] S. Boucheron, O. Bousquet, G. Lugosi, P. Massart, *Moment inequalities for functions of independent random variables*, Ann. Probab. 33 (2005), 514–560.

[Bu88] D. L. Burkholder, *Sharp inequalities for martingales and stochastic integrals*, Astérisque 157/158 (1988), 75–94.

[BG70] D. L. Burkholder, R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124 (1970), 249–304.

[Eg07] V. A. Egorov, *Estimation of constants in the right-hand side of the inequalities of Marcinkiewicz and Rosenthal*, Zap. Nauchn. Sem. POMI 341 (2007), 115–123; English transl.: J. Math. Sci. (N.Y.) 147 (2007), 6912–6917.

[Eg09] V. A. Egorov, *Two-sided estimates for constants in the Marcinkiewicz inequalities*, Zap. Nauchn. Sem. POMI 361 (2008), 45–56; English transl.: J. Math. Sci. (N.Y.) 159 (2009), 305–311 [estimation of the constant B_p^* in (20)].

[Gu67] R. F. Gundy, *The martingale version of a theorem of Marcinkiewicz and Zygmund*, Ann. Math. Statist. 38 (1967), 725–734.

[Ho63] W. Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. 58 (1963), 13–30.

[Ho97] P. Holgate, *Studies in the history of probability and statistics XLV. The late Philip Holgate's paper 'Independent functions: Probability and analysis in Poland between the Wars'*, Biometrika 84 (1997), 159–173.

[IS99] R. Ibragimov, Sh. Sharakhmetov, *Analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal inequalities for symmetric statistics*, Scand. J. Statist. 26 (1999), 621–633.

[JS88] W. B. Johnson, G. Schechtman, *Martingale inequalities in rearrangement invariant function spaces*, Israel J. Math. 64 (1988), 267–275.

[JSZ85] W. B. Johnson, G. Schechtman, J. Zinn, *Best constants in moment inequalities for linear combinations of independent and exchangeable random variables*, Ann. Probab. 13 (1985), 234–253.

[Ka77] N. J. Kalton, *Orlicz sequence spaces without local convexity*, Math. Proc. Cambridge Philos. Soc. 81 (1977), 253–277.

[Kh23] A. Khintchine, *Über dyadische Brüche*, Math. Z. 18 (1923), 109–116.

[Ko29] A. Kolmogoroff, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. 101 (1929), 126–135.

[Kr02] A. Krantsberg, *Khinchin and Marcinkiewicz–Zygmund-type inequalities for quadratic forms of independent random variables*, Statist. Probab. Lett. 60 (2002), 321–327 [Marcinkiewicz–Zygmund inequalities (18), (17): pages 321–322].

[Ma89] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminários de Matemática 5, Universidade Estadual de Campinas, Campinas, 1989.

[Mo79] J. Mogyoródi, *On an inequality of Marcinkiewicz and Zygmund*, Publ. Math. Debrecen 26 (1979), 267–274.

[Na00] S. V. Nagaev, *Probabilistic and moment inequalities for dependent random variables*, Teor. Veroyatnost. i Primenen. 45 (2000), no. 1, 194–202; English transl.: Theory Probab. Appl. 45 (2001), 152–160.

[PZ32] R. E. A. C. Paley, A. Zygmund, *On some series of functions, III*, Math. Proc. Cambridge Philos. Soc. 28 (1932), 190–205.

[RL01] Y.-F. Ren, H.-Y. Liang, *On the best constant in Marcinkiewicz–Zygmund inequality*, Statist. Probab. Lett. 53 (2001), 227–233.

[Ri09] E. Rio, *Moment inequalities for sums of dependent random variables under projective conditions*, J. Theoret. Probab. 22 (2009), 146–163.

[Ro70] H. P. Rosenthal, *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.

[Ts51] T. Tsuchikura, *Notes on Fourier analysis. XL. Remark on the Rademacher system*, Proc. Japan Acad. 27 (1951), 141–145.

4.2.2. *Marcinkiewicz–Zygmund strong law of large numbers and random series.* In the paper [MZ37c] on independent random variables Marcinkiewicz and Zygmund generalized the classical Kolmogorov strong law of large numbers (1933) on any $p \in (0, 2)$

(Kolmogorov proved it for $p = 1$): *Let* $(X_n)_{n\geq 1}$ *be a sequence of independent random variables with the same distribution. Let* $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. *Then* $\frac{S_n - nc}{n} \to 0$ *almost surely* (*i.e. with probability* 1 *or almost everywhere*) *for some* $c \in \mathbb{R}$ *if and only if* $E|X_1| < \infty$, *in which case* $c = EX_1$.

THEOREM 8 (Marcinkiewicz–Zygmund strong law of large numbers 1937). *Let* $0 < p < 2$ *and* $X_1, X_2, \ldots$ *be a sequence of independent random variables with the same distribution.*

(a) *If* $E|X_1|^p < \infty$, *then* $\frac{S_n - nc}{n^{1/p}} \to 0$ *almost surely, where* $c = EX_1$ *for* $1 \leq p < 2$ *and any* $c \in \mathbb{R}$ *for* $0 < p < 1$.
(b) *If* $\frac{S_n - nc}{n^{1/p}} \to 0$ *with probability* 1 *for some* $c \in \mathbb{R}$, *then* $E|X_1|^p < \infty$.

The classical Marcinkiewicz–Zygmund theorem appeared in the following monographs:

[Lo63] M. Loève, *Probability Theory*, Third Edition, Van Nostrand, Princeton, 1963 [Kolmogorov: $p = 1$; Marcinkiewicz: $p \neq 1$, pp. 242–243].

[St74] W. F. Stout, *Almost Sure Convergence*, Probability and Math. Statistics 24, Academic Press, New York, 1974 [Theorem 3.2.3 is due to Marcinkiewicz, pp. 126–128].

[LT91] M. Ledoux, M. Talagrand, *Probability in Banach Spaces, Isoperimetry and Processes*, Ergeb. Math. Grenzgeb. (3) 23, Springer, Berlin, 1991 [Marcinkiewicz–Zygmund theorem on random variables with values in Banach space B, Theorem 7.9, pp. 186–190; relation between type of a Banach space B and Marcinkiewicz–Zygmund theorem on random variables with values in B, Theorem 7.9, Theorem 9.21, pp. 259–260].

[Ka97] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York, 1997 [Theorem 3.23. Marcinkiewicz and Zygmund with proof, p. 51].

[Gu05] A. Gut, *Probability: A Graduate Course*, Springer Texts Stat., Springer, New York, 2005 [6.7. The Marcinkiewicz–Zygmund strong law and Theorem 7.1. The Marcinkiewicz–Zygmund strong law with proof, pp. 298–301].

[ALa06] K. B. Athreya, S. N. Lahiri, *Measure Theory and Probability Theory*, Springer Texts Stat., Springer, New York, 2006 [8.4. Kolmogorov and Marcinkiewicz–Zygmund SLLNs; Theorem 8.4.4 (Marcinkiewicz–Zygmund SLLNs) with proof].

[BW07] R. Bhattacharya, E. C. Waymire, *A Basic Course in Probability Theory*, Springer, New York, 2007 [Theorem 9.5 of Marcinkiewicz and Zygmund (1937) with the proof, pp. 124–126].

Theorem 8 appeared also in the following papers:

[HL92] E. Hensz, A. Łuczak, *Marcinkiewicz strong law of large numbers*, in: Probability and Mechanics in the Historical Sketches, Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), ed. Stanisław Fudali, Part 1. Probability, Szczecin, 1992, 219–222 (Polish).

[MP10] I. K. Matsak, A. M. Plichko, *On the Marcinkiewicz–Zygmund law of large numbers in Banach lattices*, Ukraïn. Mat. Zh. 62 (2010), 504–513; English transl.: Ukrainian Math. J. 62 (2010), 575–587.

[Ne98] D. Neuenschwander, *The Marcinkiewicz–Zygmund law of large numbers on the group of Euclidean motions and the diamond group*, J. Math. Sci. (New York) 89 (1998), 1535–1540.

[Sz10] Z. S. Szewczak, *Marcinkiewicz laws with infinite moments*, Acta Math. Hungar. 127 (2010), 64–84.

[Sz11] Z. S. Szewczak, *On Marcinkiewicz–Zygmund laws*, J. Math. Anal. Appl. 375 (2011), 738–744.

[Sz92] D. Szynal, *History of strong law of large numbers until 1939*, in: Probability and Mechanics in the Historical Sketches, Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), ed. Stanisław Fudali, Part 1. Probability, Szczecin, 1992, 120–176 (Polish).

The strong law of large numbers of Marcinkiewicz–Zygmund was generalized to random variables with values in Banach spaces and is closely connected with Rademacher type of a Banach space. For $p \in [1,2]$ one says that a Banach space B has *Rademacher type* p if there is some constant $C > 0$ such that

$$\Big(\int_0^1 \Big\|\sum_{k=1}^n x_k r_k(t)\Big\|_B^p\,dt\Big)^{1/p} \le C\Big(\sum_{k=1}^n \|x_k\|_B^p\Big)^{1/p}$$

for all $x_1, x_2, \ldots, x_n \in B$ and any $n \in \mathbb{N}$. Any Banach space has type 1; Hilbert spaces have type 2; for $p < q$, type q implies type p. Beck [Be62] found a necessary and sufficient condition (Beck convexity, or "B-convexity") on the geometry of B that for every B-valued independent random variables $X_1, X_2, \ldots$ with mean zero and $\sup_n E\|X_n\|_B^2 < \infty$ we have $\frac{X_1+X_2+\ldots+X_n}{n} \to 0$ as $n \to \infty$. The B-convexity condition holds if and only if B has type p for some $p > 1$.

The following extension of the Marcinkiewicz–Zygmund SLLN to B-valued random variables was proved by de Acosta [Ac81, p. 160] (see also Azlarov and Volodin [AV81]): *Let $1 \le p < 2$. A Banach space B has the Rademacher type p if and only if for every B-valued sequence of identically distributed independent random variables $(X_k)_{k=1}^\infty$ with mean zero and $E\|X_1\|_B^p < \infty$ satisfies the SLLN: $\lim_{n\to\infty} \frac{X_1+X_2+\ldots+X_n}{n^{1/p}} = 0$ almost surely.*

The Marcinkiewicz–Zygmund law of large numbers was examined by Marcus and Woyczyński [MW79] in Banach spaces of stable type, Woyczyński [Wo80], Korzeniowski [Ko84], Bingham [Bi86] and many others. More information can be found in the papers collected below and in the mentioned monograph by Ledoux and Talagrand (1991).

[Ac81] A. de Acosta, *Inequalities for B-valued random vectors with applications to the strong law of large numbers*, Ann. Probab. 9 (1981), 157–161.

[AV81] T. A. Azlarov, N. A. Volodin, *The laws of large numbers for identically distributed Banach space valued random variables*, Teor. Veroyatnost. i Primenen. 26 (1981), 584–590; English transl.: Theory Prob. Appl. 26 (1981), 573–580.

[Be62] A. Beck, *A convexity condition in Banach spaces and the strong law of large numbers*, Proc. Amer. Math. Soc. 13 (1962), 329–334.

[Bi86] N. H. Bingham, *Extensions to the strong law*, Adv. in Appl. Probab. 1986, suppl., 27–36 [Marcinkiewicz–Zygmund LLN, p. 29].

[GZ92] E. Giné, J. Zinn, *Marcinkiewicz type law of large numbers and convergence of moments for U-statistics*, in: Probability in Banach Spaces 8 (Brunswick, ME, 1991), Progr. Probab. 30, Birkhaüser, Boston, 1992, 273–291.

[HH10] F. Hechner, B. Heinkel, *The Marcinkiewicz–Zygmund LLN in Banach spaces: a generalized martingale approach*, J. Theor. Probab. 23 (2010), 509–522.

[Ko84] A. Korzeniowski, *On Marcinkiewicz SLLN in Banach spaces*, Ann. Probab. 12 (1984), 279–280.

[MW79] M. B. Marcus, W. A. Woyczyński, *Stable measures and central limit theorem in spaces of stable type*, Trans. Amer. Math. Soc. 251 (1979), 71–102.

[Ri95] E. Rio, *A maximal inequality and dependent Marcinkiewicz–Zygmund strong laws*, Ann. Probab. 23 (1995), 918–937.

[ST92] K. L. Su, R. L. Taylor, *Marcinkiewicz strong laws of large numbers and convergence rates for arrays of independent random elements in Banach spaces*, Stochastic Anal. Appl. 10 (1992), 223–237.

[Su93] Z. G. Su, *Marcinkiewicz laws of large numbers for a sequence of independent Banach space-valued random variables*, Acta Math. Sinica 36 (1993), 731–739 (Chinese).

[Su96] Z. Su, *The law of the iterated logarithm and Marcinkiewicz law of large numbers for B-valued U-statistics*, J. Theoret. Probab. 9 (1996), 679–701.

[Wo80] W. A. Woyczyński, *On Marcinkiewicz–Zygmund laws of large numbers in Banach spaces and related rates of convergence*, Probab. Math. Statist. 1 (1980), 117–131.

A frequent method of proof of the strong law of large numbers is to demonstrate the convergence almost surely of some random series and to use the Kronecker lemma. Sufficient conditions or criteria of convergence were given by Khintchine and Kolmogorov (1925). If $(X_n)_{n=1}^\infty$ is a sequence of independent random variables, then by zero-one law of Kolmogorov the probability that the series $\sum_{n=1}^\infty X_n$ is convergent is equal either to 0 or 1.

Marcinkiewicz and Zygmund in the paper [MZ37c] proved also the following theorem on random series:

Theorem 9 (Marcinkiewicz–Zygmund 1937).

(a) *Let $(X_n)_{n=1}^\infty$ be a sequence of independent random variables such that $EX_n = 0$, $E(X_n^2) = 1$, $n \geq 1$ and $\inf_{n\in\mathbb{N}} E|X_n| > 0$. If the series $\sum_{n=1}^\infty a_n X_n$ is almost surely convergent for a sequence $(a_n)_{n=1}^\infty$ of real numbers, then $\sum_{n=1}^\infty a_n^2 < \infty$.*

(b) *If $(X_n)_{n=1}^\infty$ is a sequence of independent random variables with the same distribution such that $E|X_1|^p < \infty$ for some $0 < p < 2$, then*

$$\sum_{n=1}^\infty \frac{X_n - EY_n}{n^{1/p}} < \infty \quad \textit{almost surely,}$$

where $Y_n = X_n \mathrm{I}_{\{|X_n|\leq n^{1/p}\}}$. Morever, if either $0 < p < 1$ or $1 < p < 2$ and $EX_1 = 0$, then the series $\sum_{n=1}^\infty \frac{X_n}{n^{1/p}}$ is convergent almost surely.

In 1938 Marcinkiewicz and Zygmund proved in the paper [MZ38a] the next famous result for real-valued random variables (see Theorem 10). Kahane in the book [Ka85] generalized this result to random variables with values in a Banach space X.

Assume that the infinite matrix of numbers $T = (a_{nm})_{n,m=1}^\infty$ is a summability matrix (Toeplitz matrix), that is, it satisfies the condition $\lim_{n\to\infty} a_{nm} = 1$ for $m = 1, 2, \ldots$. A sequence $x = (x_n)$, $x_n \in X$ is T-summable, if the series $T_n(x) = \sum_{m=1}^\infty a_{nm} x_m$ are convergent in X for each $n \in \mathbb{N}$ and the sequence $(T_n(x))$ is convergent in X.

Theorem 10 (Marcinkiewicz–Zygmund 1938). *Let $(X_n)_{n=1}^\infty$ be a sequence of independent random variables with values in a Banach space X and let T be a matrix of summability. If the series $\sum_{n=1}^\infty X_n$ is almost surely T-summable, then there exists a sequence $x_n \in X$ such that the series $\sum_{n=1}^\infty (X_n - x_n)$ is convergent almost surely. If the series $\sum_{n=1}^\infty X_n$*

is almost surely T-bounded, then there exists a sequence $x_n \in X$ such that the series $\sum_{n=1}^{\infty}(X_n - x_n)$ is bounded almost surely.

Another proof of Theorem 10 together with necessity of conditions in this theorem on convergence for the case $X = \mathbb{R}$ was given by Tucker [Tu65]. Kahane says in [Ka85] that P. Lévy has such a theorem in his paper [Le35], which I cannot see. The formulation maybe is in Lévy's book [Le37].

On almost sure convergence of random series we can read in many books. For example, the books listed below contain Marcinkiewicz–Zygmund theorems (Theorem 9: [CT88], [KS84], [St74] and Theorem 10: [Ka63], [Ka85]):

[Le37] P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris, 1937.

[Ka63] J.-P. Kahane, *Séries de Fourier aléatoires*, Séminaire de Math. Supérieures, No. 4 (1963), Université de Montréal, Montreal, 1967 [Theorem 5, pp. 39–41].

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Academic Press, New York–London, 1972 [13.7.1 and 13.8.1. Marcinkiewicz–Zygmund, pp. 583–584 and 588–590].

[KS84] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (Russian).

[Ka85] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Stud. Adv. Math. 5, Cambridge Univ. Press, Cambridge, 1985. [Theorem 2, pp. 13–17].

[CT88] Y. S. Chow, H. Teicher, *Probability Theory. Independence, Interchangeability, Martingales*, 2nd ed., Springer, New York, 1988 [Theorem 3 (Marcinkiewicz–Zygmund), p. 118].

[LW83] T. L. Lai, C. Z. Wei, *A note on martingale difference sequences satisfying the local Marcinkiewicz–Zygmund condition*, Bull. Inst. Math. Acad. Sinica 11 (1983), 1–13.

[Le35] P. Lévy, *Sur la sommabilité des séries aléatoires divergentes*, Bull. Soc. Math. France 63 (1935), 1–35.

[Tu65] H. G. Tucker, *On quasi-convergence of series of independent random variables*, Proc. Amer. Math. Soc. 16 (1965), 435–439.

4.2.3. *Law of the iterated logarithm* (1937). In 1929 Kolmogorov proved the so-called law of the iterated logarithm: *Let (X_n) be a sequence of independent random variables, each with mean zero and finite variance. Let $S_n = \sum_{k=1}^{n} X_k$ and $B_n = \sum_{k=1}^{n} E(X_k^2) \to \infty$, when $n \to \infty$. If there exists a sequence $(M_n)_{n=1}^{\infty}$ of positive numbers such that*

$$|X_n| \leq M_n \quad \textit{and} \quad M_n = o\bigg(\Big(\frac{B_n}{\log\log B_n}\Big)^{1/2}\bigg), \tag{25}$$

i.e. $X_n = o\big((\frac{B_n}{\log\log B_n})^{1/2}\big)$ almost surely, then

$$\mathrm{P}\Big(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\Big) = 1, \tag{26}$$

that is, $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1$ almost surely.

The second assumption in condition (25) cannot be weakened, which was proved by Marcinkiewicz and Zygmund in their paper [MZ37e] from 1937.

Theorem 11 (Marcinkiewicz–Zygmund 1937). *There exists a sequence $(X_n)_{n=1}^{\infty}$ of independent (two-valued) random variables such that $EX_k = 0$, $\sigma^2 X_k < \infty$ for $k \geq 1$*

and

$$M_n := \max_{1\le k\le n} |X_k| = O\Big(\Big(\frac{B_n}{\log\log B_n}\Big)^{1/2}\Big) \quad \textit{and} \quad \mathrm{P}\Big(\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = 1\Big) = 0,$$

i.e., $\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} < 1$ *almost surely.*

It is worth to mention a remark of Marcinkiewicz to Zygmund, which is not in the joint paper with Zygmund (only in the overview of Marcinkiewicz's results [Zy60], p. 38): for any sequence of numbers $(a_n)_{n=1}^{\infty}$ such that $B_n = \sum_{k=1}^{n} a_k \to \infty$ as $n \to \infty$ we have

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k\, r_k(t)}{\sqrt{2B_n\ \log\log B_n}} \le 1$$

for almost all points t from the interval $[0,1]$. The proof follows from classical argument of Kolmogorov.

Let us note that Hartman and Wintner [HW41] showed in 1941 that if $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same distribution such that $E(X_1) = \mu$ and $\sigma^2(X_1) = \sigma^2 < \infty$, then

$$\mathrm{P}\Big(\liminf_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = -1\Big) = \mathrm{P}\Big(\limsup_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = 1\Big) = 1.$$

From here it can be derived that with probability 1 the set of all limit points of the sequence $(\frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}})_{n=3}^{\infty}$ is the interval $[-1,1]$.

Strassen [St66] proved a converse theorem to the law of iterated logarithm: if $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables with the same distribution and

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1,$$

then $EX_1 = 0$ and $EX_1^2 = 1$.

A survey article on law of the iterated logarithm was published by Bingham [Bi86a]. Moreover, these problems appeared in the following books and papers:

[St74] W. F. Stout, *Almost Sure Convergence*, Probability and Math. Statistics 24, Academic Press, New York, 1974 [Part 5.2].

[Pe95] V. V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford Stud. Probab. 4, Oxford Univ. Press, New York, 1995.

[Bi86a] N. H. Bingham, *Variants on the law of the iterated logarithm*, Bull. London Math. Soc. 18 (1986), 433–467.

[Bi00] N. H. Bingham, *Studies in the history of probability and statistics XLVI. Measure into probability: from Lebesgue to Kolmogorov*, Biometrika 87 (2000), 145–156.

[Fe45] W. Feller, *The fundamental limit theorems in probability*, Bull. Amer. Math. Soc. 51 (1945), 800–832.

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[Ha41] P. Hartman, *Normal distributions and the law of the iterated logarithm*, Amer. J. Math. 63 (1941), 584–588.

[HW41] P. Hartman, A. Wintner, *On the law of the iterated logarithm*, Amer. J. Math. 63 (1941), 169–176.

[Ko29] A. Kolmogoroff, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. 101 (1929), 126–135.

[St66] V. Strassen, *A converse to the law of the iterated logarithm*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete 4 (1966), 265–268.

[Sz92] D. Szynal, *History of strong law of large numbers until 1939*, in: Probability and Mechanics in the Historical Sketches, Proc. of the 5th All-Polish School on the History of Mathematics (Dziwnów, 9–13 May 1991), ed. Stanisław Fudali, Part 1. Probability, Szczecin, 1992, 120–176 (Polish).

[To71] R. J. Tomkins, *Some iterated logarithm results related to the central limit theorem*, Trans. Amer. Math. Soc. 156 (1971), 185–192.

After Kolmogorov law of iterated logarithm different forms of this theorem became an object of interest of several mathematicians (see Feller [Fe43] and the references given there). Some bounds for the sums $S_n = \sum_{k=1}^{n} X_k$ of a sequence of independent random variables $(X_k)_{k=1}^{\infty}$ were proved in 1931 by Lévy [Le31] for $0 < \alpha < 1$ using the stable distribution. The method does not work for $\alpha \geq 1$ and he formulated the question if his result is also true for $1 \leq \alpha < 2$. Marcinkiewicz [M39k] gave a positive answer in 1939. Afterwards this result was named the *Lévy–Marcinkiewicz theorem* (cf. Feller [Fe46], pp. 257–258): *Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables. Suppose that one has estimates uniformly for large u and all $k \in \mathbb{N}$*

$$cu^{-\alpha} \leq P(|X_k| > u) \leq Cu^{-\alpha},$$

where c, C are positive constants and $0 < \alpha < 1$. Let $\lambda(t)$ be an increasing function such that $\lim_{t\to\infty} \frac{\lambda(2t)}{\lambda(t)} = 1$. Then the probability for an infinite number of realizations of the inequality

$$|S_n| > n \log n \lambda(\log n)^{1/\alpha}$$

is zero (one) if the series $\sum_{n=1}^{\infty} \frac{1}{n\lambda(n)}$ converges (diverges). The theorem remains true also for $1 \leq \alpha < 2$ provided that $E(X_k) = 0$.

Some generalizations of the Lévy–Marcinkiewicz result were given by Feller [Fe43], [Fe45], [Fe46], Kunisawa [Ku49b], [Ku49], Lipschutz [Li56] and later generalizations obtained by many authors received the name of Feller.

[Ku49b] K. Kunisawa, *Limit Theorems in Probability Theory*, Chûbunkan, Tokyo, 1949 (Japanese) [Chapter 10, generalization of the Lévy–Marcinkiewicz theorem].

[Fe43] W. Feller, *The general form of the so-called law of the iterated logarithm*, Trans. Amer. Math. Soc. 54 (1943), 373–402 [Lévy–Marcinkiewicz result, p. 377].

[Fe45] W. Feller, *The fundamental limit theorems in probability*, Bull. Amer. Math. Soc. 51 (1945), 800–832 [Lévy and Marcinkiewicz result, p. 809].

[Fe46] W. Feller, *A limit theorem for random variables with infinite moments*, Amer. J. Math. 68 (1946), 257–262 [Lévy–Marcinkiewicz theorem, p. 257–258].

[Ku49] K. Kunisawa, *On an analytical method in the theory of independent random variables*, Ann. Inst. Statist. Math., Tokyo 1 (1949), 1–77 [6.3.2. Lévy–Marcinkiewicz theorem, p. 75].

[Le31] P. Lévy, *Sur les séries dont les termes sont des variables éventuelles indépendantes*, Studia Math. 3 (1931), 117–155.

[Li56] M. Lipschutz, *On strong bounds for sums of independent random variables which tend to a stable distribution*, Trans. Amer. Math. Soc. 81 (1956), 135–154 [Marcinkiewicz and Lévy result, p. 136].

4.2.4. *Marcinkiewicz's theorem on characteristic function* (1938). Let X be a real-valued random variable on a probability space (Ω, Σ, P) with the distribution function $F(x)$. The *characteristic function* (or the Fourier–Stieltjes transform) of the random variable X (or of the distribution function F) is a function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) = E(e^{itX}) = \int_{\mathbb{R}} e^{itx}\, dF(x), \quad t \in \mathbb{R}.$$

There is interest to decide whether a given function $f(t)$ can be a characteristic function, i.e., whether it admits the above representation. Necessary and sufficient conditions are known which a complex-valued function of a real variable t must satisfy in order to be a characteristic function of some random variable. The central result here is Bochner's theorem (1932), although its usefulness is limited because the main condition of the theorem, positive definiteness, is very hard to verify. Other theorems also exist, such as Mathias (1923), Khintchine (1937), or Cramér (1937), although their application is just as difficult (cf. Linnik [Li64], pp. 41–48). Pólya's theorem (1949), on the other hand, provides a very simple convexity condition which is sufficient but not necessary. However, these general conditions are not easily applicable. Therefore various conditions were derived which are restricted to certain classes of functions but are applied more readily.

A characteristic function f is said to be an *analytic characteristic function* if it coincides in $-\delta < t < \delta$ for some $\delta > 0$ with a function of complex variable $z = t + iv$ which is analytic in the disc $|z| < \delta$.

Marcinkiewicz, in the paper [M39a] from 1939, showed that the exponential function with the base e and exponent given by polynomial of degree higher than 2 is not a characteristic function of any random variable, that is, if $\varphi(t) = \exp(P(t))$, where P is a polynomial such that $P(0) = 0$ is a characteristic function of some random variable X, then $P(t)$ is a polynomial of degree 2 and X is a Gaussian variable.

Theorem 12 (Marcinkiewicz's theorem on characteristic function 1938). *If a polynomial $P(t) = P_n(t) = \sum_{k=1}^{n} a_k t^k$ is of degree $n > 2$, then the function $\varphi(t) = \exp(P(t))$ is not a characteristic function. More general, any entire function of finite order ρ, which convergence exponent is smaller than ρ cannot be a characteristic function.*

In other words Marcinkiewicz's theorem asserts: No function of the form $\exp(\sum_{k=1}^{n} a_k z^k)$ with $n > 2$ can be a characteristic function; also if the function $\varphi(t) = \exp[P(t)]$ with $P(t) = \sum_{k=1}^{n} a_k t^k$, $a_k \in \mathbb{C}$, is a characteristic function, then either $P(t) = -at^2 + ibt$, $a > 0$, $b \in \mathbb{C}$ (Gaussian law) or $P(t) = ibt$ (degenerate law).

Marcinkiewicz's theorem was extended to iterated exponents and certain functions of the form $f(t) = g(t) \exp[P(t)]$ by Lukacs [Lu58]: If

$$e_1(z) = \exp(z),\ e_2(z) = e^{e_1(z)},\ \ldots,\ e_k(z) = e^{e_{k-1}(z)}$$

and $P_m(t) = \sum_{k=0}^{m} c_k t^k$ is a polynomial of degree $m > 2$, then for any $n \geq 1$ the function $f_n(t) = c_n e_n[P_m t)]$ with constant c_n determined by the condition that $f_n(0) = 1$ cannot be a characteristic function (for $n = 1$ this is the Marcinkiewicz theorem).

Further extension was done by Christensen [Ch62] to certain functions of the form $f_n(t) = c_n g(t) e_n[P_m(t)]$, where $g(t)$ is some specified characteristic function. Cairoli [Ca64] investigated similar problems for meromorphic functions of finite order. Miller

[Mi67] studied entire functions of the form $g(t)\{\exp[P(t)]\}$ or $f\{\exp[P(t)]\}$, where $g(t)$ and $f(t)$ are entire functions while $P(t)$ is a polynomial.

Important generalizations of Marcinkiewicz's theorem were obtained by Ostrovskiĭ (1962, 1966, 1983). In 1962 Ostrovskiĭ [Os62] proved a conjecture of Linnik on strengthened Marcinkiewicz theorem for entire characteristic functions without zeros (he gave a simpler proof in [Os83]): If a characteristic function $f(t)$ has the form $f(t) = \exp[g(t)]$, where $g(t)$ is entire function such that $\log^+ \max_{|z|=r} |f(z)| = o(r)$ as $r \to \infty$, then $f(t)$ is the characteristic function of the Gaussian law.

Marcinkiewicz's theorem was further generalized by Sapogov (1976, 1979), Kamynin (1979), Golinskiĭ (1986, 1988) and Feldman (1989) on other classes of analytic functions, and also on distribution functions of several variables by Rajagopal and Sudarshan (1974) as well as on matrix-valued analytic characteristic functions by Gyires (1983).

Marcinkiewicz's theorem is useful and used by many authors in studies concerning the characterization of the normal distribution. For example, we can find it in the books by Linnik (1964), Lukacs and Laha (1964), Ramachandran (1967), Lukacs (1970), Kagan, Linnik and Rao (1973), Linnik and Ostrovskiĭ (1977), Bryc (1995) and Feldman (1990, 2008).

Among the books and papers thematically related to Theorem 12 it is worth to mention the following ones:

[Li64] Yu. V. Linnik, *Decomposition of Probability Distributions*, Oliver & Boyd, Edinburgh-London, 1964 [3.3.1. Marcinkiewicz theorem, pp. 56–58]; Russian version: Izdat. Leningrad. Univ., Leningrad, 1960.

[LL64] E. Lukacs, R. G. Laha, *Applications of Characteristic Functions*, Hafner, New York, 1964 [Marcinkiewicz' theorem (Lemma 5.1.2), p. 75].

[Ra67] B. Ramachandran, *Advanced Theory of Characteristic Functions*, Statistical Publishing Soc., Calcutta, 1967 [3.13 and 3.14. Marcinkiewicz theorems, pp. 63–64].

[Lu70] E. Lukacs, *Characteristic functions*, 2nd ed., Hafner, New York, 1970 [Corollary to theorem 7.3.3 (Theorem of Marcinkiewicz), p. 213; Theorem 7.3.4 – theorem of Marcinkiewicz, pp. 221–225].

[KLR73] A. M. Kagan, Yu. V. Linnik, C. R. Rao, *Characterization Problems in Mathematical Statistics*, Wiley, New York–London–Sydney, 1973 [Lemma 1.4.2. Marcinkiewicz' theorem, p. 25]; Russian version: Nauka, Moscow, 1972.

[LO77] Ju. V. Linnik, I. V. Ostrovskiĭ, *Decomposition of Random Variables and Vectors*, Transl. Math. Monogr. 48, Amer. Math. Soc., Providence, 1977 [II. 5. Marcinkiewicz's theorem, pp. 41–42 and 361]; Russian version: Nauka, Moscow, 1972.

[Fe90] G. M. Feldman, *Arithmetic of Probability Distributions, and Characterization Problems on Abelian Groups*, Naukova Dumka, Kiev, 1990 (Russian); English transl.: Transl. Math. Monogr. 116, Amer. Math. Soc., Providence, 1993 [Appendix 1. Group analogs of the Marcinkiewicz theorem and the Lukacs theorem, pp. 173–177].

[Br95] W. Bryc, *The Normal Distribution. Characterizations with Applications*, Lecture Notes in Statist. 100, Springer, New York, 1995 [in Section 2.5 two classical theorems appeared, Cramér decomposition theorem and Marcinkiewicz theorem, giving criteria for normality].

[Fe08] G. Feldman, *Functional Equations and Characterization Problems on Locally Compact Abelian Groups*, EMS Tracts in Math. 5, European Math. Soc., Zürich, 2008 [II.5. Polynomials on locally compact Abelian groups and the Marcinkiewicz theorem, pp. 38–49].

[Ca64] R. Cairoli, *Sur les fonctions caractéristiques de lois de probabilité*, Publ. Inst. Statist. Univ. Paris 13 (1964), 45–53.

[Ch62] I. F. Christensen, *Some further extensions of a theorem of Marcinkiewicz*, Pacific J. Math. 12 (1962), 59–67.

[Fe89] G. M. Feldman, *Marcinkiewicz and Lukacs theorems on abelian groups*, Teor. Veroyatnost. i Primenen. 34 (1989), 330–339; English transl.: Theory Probab. Appl. 34 (1989), 290–297.

[Go86] L. B. Golinskiĭ, *An estimate for stability in the Marcinkiewicz theorem for fourth-degree polynomials*, in: Mathematical Physics, Functional Analysis, Naukova Dumka, Kiev, 1986, 118–126 (Russian).

[Go88] L. B. Golinskiĭ, *Stability estimates in a theorem of J. Marcinkiewicz*, in: Stability Problems for Stochastic Models, (Sukhumi, 1987), VNIISI, Moscow, 1988, 8–24; English transl.: J. Soviet Math. 57 (1991), 3193–3209.

[Gy83] B. Gyires, *On matrix-valued analytic characteristic functions*, Publ. Math. Debrecen 30 (1983), 133–142.

[Ka79] I. P. Kamynin, *Generalization of the theorem of Marcinkiewicz on entire characteristic functions of probability distributions*, Zap. Nauch. Sem. Leningrad. Otd. Mat. Inst. Steklov. (LOMI) 85 (1979), 94–103; English transl.: J. Math. Sci. 20 (1982), 2175–2180.

[Lu58] E. Lukacs, *Some extensions of a theorem of Marcinkiewicz*, Pacific J. Math. 8 (1958), 487–501.

[Lu72] E. Lukacs, *A survey of the theory of characteristic functions*, Advances in Appl. Probability 4 (1972), 1–38 [Theorem 3.6 (Marcinkiewicz), p. 14].

[Mi67] H. D. Miller, *Generalization of a theorem of Marcinkiewicz*, Pacific J. Math. 20 (1967), 261–274.

[Os62] I. V. Ostrovskiĭ, *Application of a rule of Wiman and Valiron to the study of the characteristic functions of probability laws*, Dokl. Akad. Nauk SSSR 143 (1962), 532–535 (Russian).

[Os66] I. V. Ostrovskiĭ, *On the growth of entire characteristic functions of probabilistic laws*, in: Contemporary Problems in Theory Anal. Functions (Internat. Conf., Erevan, 1965), Nauka, Moscow, 1966, 239–245 (Russian).

[Os83] I. V. Ostrovskiĭ, *On the growth of entire characteristic functions*, in: Stability Problems for Stochastic Models (Moscow, 1982), Lecture Notes in Math. 982, Springer, Berlin, 1983, 151–155.

[RS74] A. K. Rajagopal, E. C. G. Sudarshan, *Some generalizations of the Marcinkiewicz theorem and its implications to certain approximation schemes in many-particle physics*, Phys. Rev. A (3) 10 (1974), 1852–1857.

[Sa76] N. A. Sapogov, *Stability for the Marcinkiewicz theorem. The case of a fourth degree polynomial*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 61 (1976), 107–124, 137–138 (Russian).

[Sa79a] N. A. Sapogov, *Weak stability of J. Marcinkiewicz's theorem and some inequalities for characteristic functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85 (1979), 193–196 (Russian).

[Sa79b] N. A. Sapogov, *The problem of stability for J. Marcinkiewicz's theorem*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 87 (1979), 104–124, 208–209 (Russian).

4.3. Real mathematical analysis. In this part we will deal with Marcinkiewicz integral, Marcinkiewicz function, theorem on strong differentiation of integrals, maximal function, strong maximal function and Marcinkiewicz decomposition being a prototype of the Calderón–Zygmund decomposition.

4.3.1. *Marcinkiewicz integral, Marcinkiewicz function and decomposition of Marcinkiewicz–Zygmund.* Examining the existence of the conjugate function and its weak type $(1,1)$ Marcinkiewicz analyzed a special function and structure of closed subsets. Let us consider the one-dimensional case. For a closed set $P \subset \mathbb{R}^1$ and a point x let

$$\delta(x) = \delta(x, P) = \inf\big\{|x-y| : y \in P)\big\}$$

denote the distance of x to P. This function satisfies the Lipschitz condition, i.e., $|\delta(x) - \delta(y)| \leq |x-y|$. Marcinkiewicz proved the following result:

Theorem 13 (Marcinkiewicz 1938).

(i) *If P is a closed subset of a bounded open interval (a,b) and $\lambda > 0$, then the integral*

$$I_\lambda(x) = I_\lambda(x;P) = \int_a^b \frac{\delta(y)^\lambda}{|x-y|^{1+\lambda}}\,dy \tag{27}$$

is finite for almost all $x \in P$. Moreover, $I_\lambda \in L^1(P)$ and $\int_P I_\lambda(x)\,dx \ \ \leq \frac{2}{\lambda} m((a,b)\setminus P)$.

(ii) *If P is a closed subset in $\mathbb{R}^1$ and f a nonnegative integrable function on $\mathbb{R}^1 \setminus P$, then the function*

$$J_\lambda(f)(x) = \int_{\mathbb{R}^1} \frac{\delta(y)^\lambda f(y)}{|x-y|^{1+\lambda}}\,dy \tag{28}$$

is integrable on P, and hence is finite almost everywhere in P.

In the limiting case $\lambda = 0$, (27) and (28) should be replaced by the integrals

$$I_0(x) = \int_a^b \frac{(\log 1/\delta(y))^{-1}}{|x-y|}\,dy, \quad J_0(f)(x) = \int_a^b \frac{f(y)\,(\log 1/\delta(y))^{-1}}{|x-y|}\,dy, \tag{29}$$

respectively. The integrals in (27), (28) and (29) are called *Marcinkiewicz integrals*. The integral (27) is discussed in the papers [M36a], [MZ36], [M38h], [M39f] and the integrals (29) in the paper [M35d]. Zygmund noted the following (cf. [Zy64], p. 5):

> *Marcinkiewicz proved this theorem in a somewhat different form by considering in* (27) *instead of the function $\delta(x)$ the function $\psi(x)$ which is equal to* 0 *in P and is equal to d in each interval contiguous to P and having length d, but the proofs in both cases are analogous, and the function δ is easier to use than ψ, especially if we consider the analogue of the theorem in the n-dimensional space. The proof of the theorem is not particularly difficult, and a discrete sum somewhat similar to the integral* (27) *in the case $\lambda = 2$ appears in an earlier paper of Besicovitch* [Be26], *in the proof of the existence of the conjugate function*; *it is possible that Marcinkiewicz knew that paper. The merit of Marcinkiewicz was that he understood the significance of the result transcending its individual application, and by using it systematically succeeded in obtaining a number of very interesting results in the theory of trigonometric series.*

In general, in $\mathbb{R}^n$, for a closed subset P in $\mathbb{R}^n$, $\lambda > 0$ and f nonnegative measurable function on $\mathbb{R}^n$, we consider the *Marcinkiewicz integral*

$$J_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda}}\, d\mathbf{y} \quad (\mathbf{x} \in \mathbb{R}^n) \tag{30}$$

and its modified forms

$$H_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda} + \delta(\mathbf{x})^{n+\lambda}}\, d\mathbf{y}, \quad H'_\lambda(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\delta(\mathbf{y})^\lambda f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\lambda} + \delta(\mathbf{y})^{n+\lambda}}\, d\mathbf{y},$$

introduced by Carleson and Zygmund. Observe that

$$2^{-n-\lambda-1}\, H'_\lambda(f)(\mathbf{x}) \le H_\lambda(f)(\mathbf{x}) \le 2^{n+\lambda+1}\, H'_\lambda(f)(\mathbf{x}).$$

The proof of the following theorem is given in ([WZ77], Theorem 9.19): *If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ and $\lambda > 0$, then $H_\lambda(f) \in L^p(\mathbb{R}^n)$ and*

$$\|H_\lambda(f)\|_p \le C\, \|f\|_p,$$

where the constant $C > 0$ is independent of f. In particular, $\|J_\lambda(f)\|_{L^p(P)} \le C\,\|f\|_p$.

The Marcinkiewicz integral was an important tool in the proof of Calderón–Zygmund estimate of weak type $(1,1)$ for n-dimensional strongly singular integrals (Hilbert integrals, cf. Calderón and Zygmund [CZ52], and Stein [St75], pp. 14–19).

Different variants and generalizations of the Marcinkiewicz integral, and boundedness, not only in L^p-spaces, were studied by Ostrow and Stein [OS57], Yano [Ya59], Zygmund [Zy69], Fefferman and Stein [FS71], Yano [Ya75], A. P. Calderón [Ca76], C. P. Calderón [Ca78], Kruglyak and Kuznetsov [KK07].

[Zy59] A. Zygmund, *Trigonometric series*, Vol. I, II, Cambridge Univ. Press, Cambridge, 1959 [IV.2. A theorem of Marcinkiewicz, pp. 129–130].

[St75] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton NJ, 1970 [2.3. Integral of Marcinkiewicz, pp. 14–15].

[WZ77] R. L. Wheeden, A. Zygmund, *Measure and Integral. An Introduction to Real Analysis*, Marcel Dekker, New York–Basel, 1977 [Theorem of Marcinkiewicz, pp. 95–96; 4. The Marcinkiewicz integral, pp. 157–159].

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[St93] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, 1993 [Marcinkiewicz integral, p. 76].

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[CZ52] A. P. Calderón, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), 85–139.

[Ca78] C. P. Calderón, *On a lemma of Marcinkiewicz*, Illinois J. Math. 22 (1978), 36–40.
[Car66] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), 135–157.
[FS71] C. Fefferman, E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.
[FS72] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193 [standard Marcinkiewicz "distance function integral", p. 190].
[KK07] N. Kruglyak, E. A. Kuznetsov, *The limiting case of the Marcinkiewicz integral: growth for convex sets*, Proc. Amer. Math. Soc. 135 (2007), 3283–3293.
[OS57] E. H. Ostrow, E. M. Stein, *A generalization of lemmas of Marcinkiewicz and Fine with applications to singular integrals*, Ann. Scuola Norm. Sup. Pisa (3) 11 (1957), 117–135.
[Ya59] S. Yano, *On a lemma of Marcinkiewicz and its applications to Fourier series*, Tôhoku Math. J. (2) 11 (1959), 191–215.
[Ya75] S. Yano, *On Marcinkiewicz integral*, Tohoku Math. J. (2) 27 (1975), 381–388.
[Zy69] A. Zygmund, *On certain lemmas of Marcinkiewicz and Carleson*, J. Approx. Theory 2 (1969), 249–257.

The Marcinkiewicz function appeared in his paper [M38h] from 1938, whose subject is on the borderline of real and complex variable. As is known, Littlewood and Paley (1936) considering the behaviour on the boundary of an analytic function $f(z)$ in the unit circle $|z| < 1$, whose real part is $f(\theta)$, introduced the function of real variable

$$g(f)(\theta) = \Big(\int_0^1 (1-r)|f'(re^{i\theta})|^2\,dr\Big)^{1/2}$$

and proved that $\|g(f)\|_p = (\int_0^{2\pi} |g(f)(\theta)|^p\,d\theta)^{1/p} \approx \|f\|_p = (\int_0^{2\pi} |f(e^{i\theta})|^p\,d\theta)^{1/p}$ for $1 < p < \infty$ if $\int_0^{2\pi} f(\theta)\,d\theta = 0$ (the equivalence constants depend only on p). Marcinkiewicz and Zygmund [MZ38c] showed also that $\|g(f)\|_p \le C_p\,\|f\|_p$ for $0 < p < \infty$. Luzin (1930) considered a function

$$s(f)(\theta) = \Big(\iint_\Omega |f'|^2\,d\omega\Big)^{1/2},$$

where Ω is a "triangle" area in $|z| < 1$ with a vertex at $e^{i\theta}$. Marcinkiewicz and Zygmund in [MZ38c] also obtained the following estimates:

$$\|s(f)\|_p \le A_p\,\|f\|_p \text{ for } 0 < p < \infty \text{ and } \|f\|_p \le B_p\,\|s(f)\|_p \text{ for } 1 < p < \infty.$$

Marcinkiewicz and some others before him were seeking for an analogue of the function g without use of complex variable. At the first moment we can think that if f is a 2π-periodic function from L^2, then the expression

$$\Big(\int_0^{2\pi} |f(\theta+t) - f(\theta-t)|^2\,\frac{dt}{t}\Big)^{1/2}$$

will be proper, but there are simple examples of continuous functions f indicating that this integral can be infinite for all θ. Marcinkiewicz accurately predicted the usefulness of the expression μ, now called *Marcinkiewicz function*, given by the formula

$$\mu(f)(x) := \Big(\int_0^{2\pi} \Big|\frac{F(\theta+t) + F(\theta-t) - 2F(\theta)}{t}\Big|^2\,\frac{dt}{t}\Big)^{1/2}, \quad F(\theta) = \int_0^\theta f(t)\,dt + C, \tag{31}$$

about which he proved the following result ([M38g], Theorem 1): *Let F be a 2π-periodic function from L^2. If F is differentiable at each point of the set of positive measure E, then the integral* (31) *is finite at almost each point of the set E.*

Marcinkiewicz considered in [M38h] for 2π-periodic functions from L^p, $p > 1$ a more general situation, namely, functions μ_r, $r \geq 1$, defined as

$$\mu_r(f)(x) := \Big(\int_0^{2\pi} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^r \frac{dt}{t}\Big)^{1/r}$$

and proved the following estimates:

$$\|\mu_p(f)\|_p \leq B_p \, \|f\|_p \text{ for } p \geq 2,$$

$$\|f\|_p \leq C_p \, \|\mu_p(f)\|_p \text{ for } 1 < p \leq 2 \text{ with } \int_0^{2\pi} f(t)\, dt = 0.$$

Marcinkiewicz raised the question if for $\mu = \mu_2$ we have

$$A_p \, \|f\|_p \leq \|\mu_2(f)\|_p \leq B_p \, \|f\|_p,$$

assuming, of course, in the first inequality that $\int_0^{2\pi} f(t)\, dt = 0$. Zygmund ([Zy44], Theorem 1, p. 184) gave a positive solution. An analogue of the Marcinkiewicz function on $\mathbb{R}$ is, for $f \in L^1$, the following function

$$\mu(f)(x) = \Big(\int_{\mathbb{R}^1} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^2 \frac{dt}{t}\Big)^{1/2} \tag{32}$$

and on $\mathbb{R}^n$

$$\mu(f)(x) = \Big(\int_{\mathbb{R}^n} \Big|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t}\Big|^2 \frac{dt}{|t|^n}\Big)^{1/2}$$

For the function (32) Waterman [Wat59] proved that if $f \in L^p(\mathbb{R}^1)$, $p > 1$, then $\|\mu(f)\|_p \approx \|f\|_p$. Definitions of g-function, s-function and Marcinkiewicz function μ in $\mathbb{R}^n$ and all the above estimates in the case of n variables were given by Stein [St58].

Investigations of the Marcinkiewicz function were and are still carried out in different directions, see e.g. [Wh69], [Wa72], [CW82], [CW83], [TW90], [SY99], [HMY07], and also [Zy44], [Ca50], [St58], [Zy59].

[Zy59] A. Zygmund, *Trigonometric series*, Vol. I, II, Cambridge Univ. Press, Cambridge, 1959 [XIV.5. The Marcinkiewicz function $\mu(\theta)$, pp. 129–130].

[Ca50] A. P. Calderón, *On a theorem of Marcinkiewicz and Zygmund*, Trans. Amer. Math. Soc. 68 (1950), 55–61.

[CW82] S. Chanillo, R. L. Wheeden, *Distribution function estimates for Marcinkiewicz integrals and differentiability*, Duke Math. J. 49 (1982), 517–619.

[CW83] S. Chanillo, R. L. Wheeden, *Relations between Peano derivatives and Marcinkiewicz integrals*, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, Ill., 1981), Wadsworth, Belmont, 1983, Vol. II, 508–525.

[HMY07] G. Hu, Y. Meng, D. Yang, *Estimates for Marcinkiewicz integrals in BMO and Campanato spaces*, Glasg. Math. J. 49 (2007), 167–187.

[LP36] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series*, Proc. London Math. Soc. (2) 42 (1936), 52–89.

[Lu30] N. Lusin, *Sur une propriété des fonctions à carré sommable*, Bull. Calcutta Math. Soc. 20 (1930), 139–154.

[SY99] W. Sakamoto, K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math. 135 (1999), 103–142.

[St58] E. M. Stein, *On the functions of Littlewood–Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. 88 (1958), 430–466.

[TW90] A. Torchinsky, S. L. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math. 60/61 (1990), 235–243.

[Wa72] T. Walsh, *On the function of Marcinkiewicz*, Studia Math. 44 (1972), 203–217.

[Wat59] D. Waterman, *On an integral of Marcinkiewicz*, Trans. Amer. Math. Soc. 91 (1959), 129–138.

[Wh69] R. L. Wheeden, *Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type*, Studia Math. 32 (1969), 73–93.

[Ya04] K. Yabuta, *Existence and boundedness of g_λ^*-function and Marcinkiewicz functions on Campanato spaces*, Sci. Math. Jpn. 59 (2004), 93–112.

[Zy44] A. Zygmund, *On certain integrals*, Trans. Amer. Math. Soc. 55 (1944), 170–204.

Marcinkiewicz and Zygmund in paper [MZ36], studying the trigonometric series, consider the so-called Riemann derivatives. A function f, defined in a neighbourhood of a point x_0, has at this point *k-th Riemann derivative* if there is a limit of the quotient $\lim_{h\to 0} \frac{\Delta_h^k f(x_0)}{(2h)^k} = D_k f(x_0)$, where the k-th difference is given by formula

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x+(k-2i)h) = f(x+kh) - \binom{k}{1} f(x+(k-2)h)$$
$$+ (-1)^2 \binom{k}{2} f(x+(k-4)h) + \ldots + (-1)^k \binom{k}{k} (f(x-kh).$$

The special case $k=2$ is the known *Schwarz derivative.* The main result in the paper ([MZ36], Thm 1) is the following:

THEOREM 14 (Marcinkiewicz–Zygmund 1936). *If at each point x_0 of a set E of positive measure the quotient $\frac{\Delta_h^k f(x_0)}{(2h)^k}$ is bounded when $h \to 0$ (in particular, if the Riemann derivative $D_k f(x_0)$ exists), then f is k-times differentiable for almost all points from E.*

The method of proof showed that for a function f of the particular form

$$f(x+h) = \sum_{j=0}^{k-1} a_j(x) \frac{h^j}{k!} + O(h^k) \quad (x \in E), \tag{33}$$

and for arbitrary $\varepsilon > 0$ there exists a set P and functions g, b such that

(i) P is a perfect set and $|E \setminus P| < \varepsilon$
(ii) $f(x) = g(x) + b(x)$, where g is of class C^k (k-th derivative is continuous) and $|b(x)| \leq C\,\delta(x,P)^k$.

In other words, the function f satisfying (33) can be decomposed on "good part" $g(x)$ and a "bad part" $b(x)$, and the bad part $b(x)$ can be nonzero only on a small set $E \setminus P$ and is estimated by the use of a Marcinkiewicz integral (27) with $\lambda = k$.

This method was an important tool of proofs in the forties. However, in 1952 Calderón and Zygmund presented their famous decomposition for functions of n variables (see [St75], Theorem 3.2) and from then this method of proof has become the leading one.

Methods used in the theory of singular Calderón–Zygmund integrals both Marcinkiewicz's decomposition and Marcinkiewicz's interpolation theorem are particularly useful and still used in various versions and variants.

[As67] J. M. Ash, *Generalizations of the Riemann derivative*, Trans. Amer. Math. Soc. 126 (1967), 181–199 [generalization of the Marcinkiewicz–Zygmund theorem].

[CZ52] A. P. Calderón, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), 85–139.

[Dy01] E. M. Dyn'kin, *Methods of the theory of singular integrals: Hilbert transform and Calderón–Zygmund theory*, in: Commutative Harmonic Analysis I, Encyclopaedia Math. Sci. 15, Springer, Berlin, 1991, 167–259 [Marcinkiewicz interpolation theorem, p. 174; Marcinkiewicz integral, p. 240].

[FW94] H. Fejzić, C. E. Weil, *Repairing the proof of a classical differentiation result*, Real Anal. Exchange 19 (1993/94), 639–643 [Marcinkiewicz–Zygmund theorem].

4.3.2. *Differentiation of integrals and maximal functions.* The classical Lebesgue theorem (1923) tells that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^1)$ and $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$ for almost all $x \in \mathbb{R}^1$ (shortly a.e.), i.e.,

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad \text{a.e.}$$

or, equivalently,

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(y)\,dy = f(x) \quad \text{a.e.}$$

An n-dimensional version has the form: if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\lim_{|Q|\to 0} \frac{1}{|Q|} \int_Q f(y)\,dy = f(x) \quad \text{for almost all } x \in \mathbb{R}^n,$$

where Q denotes an n-dimensional cube with sides parallel to the coordinate system. Even a stronger assertion is true

$$\lim_{|Q|\to 0} \frac{1}{|Q|} \int_Q |f(y) - f(x)|\,dy = 0 \quad \text{a.e. in } \mathbb{R}^n.$$

An important tool in the proof of Lebesgue (and theorems of singular integrals and convergence almost everywhere) is a maximal function of Hardy–Littlewood

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where supremum is taken over all cubes Q in $\mathbb{R}^n$ containing the point x. Note that we can equivalently analyze the maximal function, where instead of cubes Q we take n-dimensional balls $B(x,r)$ with centre at x and radius $r > 0$

$$M_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy.$$

Hardy and Littlewood [HL30] defined the maximal function M in the one-dimensional case and showed the boundedness in $L^p(\mathbb{R}^1)$ for $p > 1$ (they did not prove the weak type $(1,1)$ of M, which is a surprise). Next important step was done by F. Riesz (1932) who proved the following inequality

$$(Mf)^*(t) \le A\, f^{**}(t) = A\,\frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{for any } t>0,$$

where f^* denotes the decreasing rearrangement of $|f(x)|$. From here we are getting the weak type $(1,1)$ of the maximal function.

N. Wiener (1939) considered the maximal function Mf in the n-dimensional case and proved, with the help of the Vitali covering theorem, its important property, i.e. the weak type $(1,1)$. It is necessary to mention here that the same year Marcinkiewicz and Zygmund also investigated the two-dimensional maximal function (from which without difficulties we get the n-dimensional case for $n \ge 2$) and they proved its weak type $(1,1)$ ([MZ39b], Lemma 2, p. 551). This fact was noted only by Stein and Wainger ([SW78], p. 1245).

Theorem 15 (Wiener 1939, Marcinkiewicz–Zygmund 1939). *The maximal function M is of weak type $(1,1)$, that is, for any $f \in L^1(\mathbb{R}^n)$ and arbitrary $\lambda > 0$ the following inequality holds*

$$\big|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\big| \le \frac{B}{\lambda}\int_{\mathbb{R}^n} |f(x)|\,dx. \tag{34}$$

Wiener (1939) proved even a stronger version of the weak $(1,1)$ inequality

$$\big|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\big| \le \frac{B}{\lambda}\int_{\{x\in\mathbb{R}^n : |f(x)|>\lambda/2\}} |f(x)|\,dx \quad \forall\,\lambda>0, \tag{35}$$

and Stein [St69] showed the following reverse weak $(1,1)$ inequality for the maximal function

$$\frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} |f(x)|\,dx \le C\big|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\big| \quad \forall\,\lambda>0.$$

The classical Theorem 15 can be found in many books on analysis or harmonic analysis. We can mention as examples the books from part 4.1.1: Stein [St70, pp. 5–11], Stein and Weiss [SW71, pp. 55–56], Sadosky [Sa79, pp. 199–202], de Guzmán [Gu81, pp. 133–134], Jørsboe and Mejlbro [JM82, pp. 10–11], Kashin and Saakyan [KS84, pp. 444–445], García-Cuerva and Rubio de Francia [GR85, pp. 144–145], Torchinsky [To86, pp. 77–78], Brudnyĭ and Krugljak [BK91, pp. 79–80], Folland [Fo99, p. 96], Duoandikoetxea [Du01, p. 31], Arias de Reyna [Ar02, pp. 4–6], DiBenedetto [Di02, p. 378], Nikolski [Ni02, p. 31], Taylor [Ta06, pp. 140–141], Grafakos [Gr08, pp. 80–81], Linares and Ponce [LP09, p. 34].

Further studies of the Wiener and Stein inequalities (for arbitrary measures in $\mathbb{R}^n$) can be found, e.g., in the paper [AKMP].

Note that if we have the weak type $(1,1)$ of the maximal function, then it is not difficult to prove the Lebesgue theorem, but equivalence with the Lebesgue theorem is not so easy and it was proved by Stein in [St61].

Let us now discuss the classical theorem of Jessen, Marcinkiewicz and Zygmund on strong differentiation of integrals proved in 1935 in the paper [MJZ35]. Let the rectangle P in $\mathbb{R}^n$, $n \ge 2$, means the product of n nonempty one-dimensional intervals and let $\delta(\mathrm{P})$ be its diameter. Let also $\mathrm{P}_0 \subset \mathbb{R}^n$ be a fixed rectangle, for example, $\mathrm{P}_0 = \mathbb{R}^n$ or $\mathrm{P}_0 = \mathrm{I}^n$, $\mathrm{I} = [0,1]$ and $f \in L^1(\mathrm{P}_0)$.

We say that the integral of the function f is *strongly differentiable* at the point $x \in \mathrm{P}_0$ if the limit

$$\lim_{\delta(\mathrm{P})\to 0} \frac{1}{|\mathrm{P}|}\int_{\mathrm{P}} f(y)\,dy$$

exists and is finite, where $\mathrm{P} \subset \mathrm{P}_0$ is any rectangle containing x. This limit is called the *strong derivative* of the integral of a function f at point x.

Saks ([Sa33], pp. 231–232 and [Sa34]) and Buseman–Feller ([BF34], pp. 243–247) showed the existence of a function $f \in L^1(\mathrm{I}^2)$ whose integral is nowhere strongly differentiable. Zygmund ([Zy34], Theorem 1) however proved that for any function $f \in L^p(\mathrm{P}_0)$, $p > 1$, the strong derivative of the integral of a function f exists almost everywhere and is equal to $f(x)$.

Jessen, Marcinkiewicz and Zygmund ([MJZ35], Theorem 2) proved strong differentiability of the integral of any function $f \in L^1(\log^+ L)^{n-1}$.

Theorem 16 (Jessen–Marcinkiewicz–Zygmund 1935). *Let* $\mathrm{P}_0 \subset \mathbb{R}^n$ *be a fixed rectangle. If a function* f *is measurable and* $|f(x)|\,(\log^+|f(x)|)^{n-1}$ *is integrable on the rectangle* P_0*, then the strong derivative of the integral of the function* f *exists for almost all points in* P_0 *and is equal to* $f(x)$.

Jessen, Marcinkiewicz and Zygmund ([MJZ35], Theorem 8) also demonstrated that the statement in Theorem 16 is in some sense the best possible. Namely, let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function vanishing only at zero and having the property

$$\liminf_{u\to\infty} \frac{\Phi(u)}{u} > 0.$$

Denote by $L^{\Phi}(\mathrm{I}^n)$ the Orlicz class of all functions f on I^n such that $\Phi(|f|) \in L^1(\mathrm{I}^n)$. The authors of that work showed that if every function $f \in L^{\Phi}(\mathrm{I}^n)$ has almost everywhere strongly differentiable integral, then $f(\log^+|f|)^{n-1} \in L^1(\mathrm{I}^n)$. Thus the largest Orlicz class for which all functions have almost everywhere strongly differentiable integrals is the class L^{Φ_0} generated by the Orlicz function $\Phi_0(u) = u(\log^+ u)^{n-1}$!

The paper [MJZ35] is cited quite often, and on the conference on “Development of mathematics 1900–1950” held in Luxembourg in 1992, this paper was listed among the most important ones published in the year 1935 (cf. [DEMP], p. 22).

The maximal function appropriate to the strong differentiation is the *strong maximal function*

$$M_S f(x) := \sup_{\mathrm{P}\ni x} \frac{1}{|\mathrm{P}|}\int_{\mathrm{P}} |f(y)|\,dy,$$

where supremum is taken over all rectangles P from $\mathbb{R}^n$ containing the point x.

Note that the function M_S is not of the weak type $(1,1)$. Namely, if we take as f, for example, the characteristic function of the unit ball, then for large coordinates $x_1, x_2, \ldots, x_n$ of a point x the quantity $M_S f(x)$ is of order $(|x_1|\cdot\ldots\cdot|x_n|)^{-1}$ and the inequality for weak type $(1,1)$ cannot be true. On the other hand, $M_S f(x)$ is pointwise bounded by the composition of one-dimensional maximal functions in each coordinate independently $M_S f(x) \le M_1(M_2(\ldots(M_n f(x))\ldots)$. Each of these maximal functions M_k, $1 \le k \le n$, is bounded in L^p, $p > 1$, therefore the function M_S is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$.

To prove the Jessen–Marcinkiewicz–Zygmund theorem with the help of the strong maximal function we need a proper replacement of the inequality on week type $(1,1)$ by another estimate. An apropriate estimate was given by de Guzmán [Gu74]: *the strong maximal function satisfies for any* $\lambda > 0$ *the best possible inequality*

$$\left|\{x \in \mathbb{R}^n : M_S f(x) > \lambda\}\right| \le C_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(1 + \frac{|f(x)|}{\lambda}\Big)^{n-1} dx, \tag{36}$$

i.e., $M_S : L(1 + (\log^+ L)^{n-1})(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ *is bounded.*

A geometrical proof, with the help of the corresponding covering lemma, was given by A. Córdoba and R. Fefferman [CF75]. This proof is repeated in the appendix of the book [Gu75].

An inequality of type (36) for the strong maximal function M_s, where by definition the supremum is taken over all rectangles P contained in I^n, that is, the inequality

$$\left|\{x \in \mathrm{I}^n : M_s f(x) > 4\lambda\}\right| \le D_n \int_{\mathrm{I}^n} \frac{|f(x)|}{\lambda} \Big(\log^+ \frac{|f(x)|}{\lambda}\Big)^{n-1} dx, \tag{37}$$

was proved for $n = 2$ by Flett [Fl55] with $D_2 = 4$ and $1 + \log(\cdot)$ in place $\log^+(\cdot)$, and for $n \ge 2$ by Fava [Fa72].

Books containing the problem of differentiation of integrals are, for example, the following ones:

[Sa33] S. Saks, *Théorie de l'intégrale*, Monografje Matematyczne II, Warszawa, 1933.

[Sa37] S. Saks, *Theory of the Integral*, 2nd rev. ed., Monografie Matematyczne VII, Warszawa-Lwów, 1937.

[Gu75] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math. 481, Springer, Berlin, 1975; Russian transl.: Mir, Moscow, 1978.

[Gu81] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland Math. Stud. 46, North-Holland, Amsterdam, 1981.

There are books, where Theorem 16 is cited as the Jessen–Marcinkiewicz–Zygmund theorem:

[KoK91] V. Kokilashvili, M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific, River Edge, NJ, 1991 [Jessen, Marcinkiewicz and Zygmund result, p. 142].

[Ko07] A. Korenovskii, *Mean Oscillations and Equimeasurable Rearrangements of Functions*, Lect. Notes Unione Mat. Ital. 4, Springer, Berlin and UMI, Bologna, 2007 [Theorem 1.3 (Jessen, Marcinkiewicz and Zygmund), p. 5].

[St93] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, 1993 [pp. 76 and 661].

[Zh96] L. Zhizhiashvili, *Trigonometric Fourier Series and Their Conjugates*, Math. Appl. 372, Kluwer, Dordrecht, 1996 [Jessen, Marcinkiewicz and Zygmund result, pp. 130, 149, 152, 188, 189, 279].

Various discussion on Theorem 16, different proofs and generalizations can be found in many papers. Below are presented some publications related to the above problems, and those in which Jessen–Marcinkiewicz–Zygmund theorem is mentioned: Saks (1935), Burkill (1951), Smith (1956), Zygmund (1967), Bruckner (1971), Fava (1972), de Guzmán (1974, 1976, 1986), Cordoba and Fefferman (1975), Strömberg (1977), Stein and Wainger

(1978), Fava, Gatto and Gutiérez (1980), Bagby (1983), Soria (1986), Stokolos (1998, 2005, 2008), Kuchta, Morayne and Solecki (2001), Hagelstein (2004).

[AKMP] I. U. Asekritova, N. Ya. Krugljak, L. Maligranda, L. E. Persson, *Distribution and rearrangement estimates of the maximal function and interpolation*, Studia Math. 124 (1997), 107–132.

[Ba83] R. J. Bagby, *A note on the strong maximal function*, Proc. Amer. Math. Soc. 88 (1983), 648–650.

[Ba83a] R. J. Bagby, *Maximal functions and rearrangements: some new proofs*, Indiana Univ. Math. J. 32 (1983), 879–891.

[Br71] A. M. Bruckner, *Differentiation of integrals*, Amer. Math. Monthly 78 (1971), no. 9, Part 2, 1–51.

[Bu51] J. C. Burkill, *On the differentiability of multiple integrals*, J. London Math. Soc. 26 (1951), 244–249.

[BF34] H. Busemann, W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math. 22 (1934), 226–256.

[Ch95] L. Chevalier, *Fonction maximale forte et intégrale de Marcinkiewicz*, J. Anal. Math. 65 (1995), 161–178.

[CF75] A. Cordoba, R. Fefferman, *A geometric proof of the strong maximal theorem*, Ann. of Math. (2) 102 (1975), 95–100.

[Du07] J. Duoandikoetxea, *The Hardy–Littlewood maximal function and some of its variants*, in: Advanced Courses of Mathematical Analysis. II, World Sci. Publ., Hackensack, NJ, 2007, 37–56.

[Fa72] N. Fava, *Weak type inequalities for product operators*, Studia Math. 42 (1972), 271–288.

[FGG80] N. A. Fava, E. A. Gatto, C. Gutiérrez, *On the strong maximal function and Zygmund's class* $L(\log^+ L)^n$, Studia Math. 69 (1980), 155–158.

[Fl55] T. M. Flett, *Some remarks on a maximal theorem of Hardy and Littlewood*, Quart. J. Math., Oxford Ser. (2) 6 (1955), 275–282.

[Gu74] M. de Guzmán, *An inequality for the Hardy–Littlewood maximal operator with respect to a product of differentiation bases*, Studia Math. 49 (1974), 185–194.

[Gu76] M. de Guzmán, *Differentiation of integrals in* $\mathbb{R}^n$, in: Measure Theory (Oberwolfach, 1975), Lecture Notes in Math. 541, Springer, Berlin, 1976, 181–185.

[Gu86] M. de Guzmán, *The evolution of some ideas in the theory of differentiation of integrals*, in: Aspects of Mathematics and its Applications, North-Holland, Amsterdam, 1986, 377–385.

[Ha04] P. A. Hagelstein, *Córdoba–Fefferman collections in harmonic analysis*, Pacific J. Math. 216 (2004), 95–109.

[Ha04a] P. A. Hagelstein, *Rearrangements and the local integrability of maximal functions*, Pacific J. Math. 216 (2004), 111–126.

[HL30] G. H. Hardy, J. E. Littlewood, *A maximal inequality with function-theoretic applications*, Acta Math. 54 (1930), 81–116.

[KMS01] M. Kuchta, M. Morayne, S. Solecki, *A martingale proof of the theorem by Jessen, Marcinkiewicz and Zygmund on strong differentiation of integrals*, in: Séminaire de Probabilités XXXV, Lecture Notes in Math. 1755 Springer, Berlin, 2001, 158–161.

[Le97] A. K. Lerner, *On estimates for strong maximal functions*, Izv. Vyssh. Uchebn. Zaved. Mat. 1997, no. 7, 36–48; English transl.: Russian Math. (Iz. VUZ) 41 (1997), no. 7, 33–45.

[Sa34] S. Saks, *Remark on the differentiability of the Lebesgue indefinite integral*, Fund. Math. 22 (1934), 257–261.

[Sa35] S. Saks, *On the strong derivatives of functions of intervals*, Fund. Math. 25 (1935), 235–252.

[Sm56] K. T. Smith, *A generalization of an inequality of Hardy and Littlewood*, Canad. J. Math. 8 (1956), 157–170.

[So86] F. Soria, *Examples and counterexamples to a conjecture in the theory of differentiation of integrals*, Ann. of Math. (2) 123 (1986), 1–9.

[St61] E. M. Stein, *On limits of sequences of operators*, Ann. of Math. (2) 74 (1961), 140–170.

[St69] E. M. Stein, *Note on the class* $L\log L$, Studia Math. 32 (1969), 305–310.

[SW78] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), 1239–1295.

[St98] A. M. Stokolos, *On a problem of A. Zygmund*, Mat. Zametki 64 (1998), 749–762; English transl.: Math. Notes 64 (1998), 646–657.

[St05] A. Stokolos, *Zygmund's program: some partial solutions*, Ann. Inst. Fourier (Grenoble) 55 (2005), 1439–1453.

[St08] A. Stokolos, *Properties of the maximal operators associated with bases of rectangles in* $\mathbb{R}^3$, Proc. Edinb. Math. Soc. (2) 51 (2008), 489–494.

[St77] J.-O. Strömberg, *Weak estimates on maximal functions with rectangles in certain directions*, Ark. Mat. 15 (1977), 229–240.

[Wi39] N. Wiener, *The ergodic theorem*, Duke Math. J. 5 (1939), 1–18.

[Zy34] A. Zygmund, *On the differentiability of multiple integrals*, Fund. Math. 23 (1934), 143–149.

[Zy67] A. Zygmund, *A note on the differentiability of integrals*, Colloq. Math. 16 (1967), 199–204.

4.3.3. *The Marcinkiewicz multiplier theorem and Marcinkiewicz sets.* Suppose there is given a Fourier series $f(x) \sim \sum c_n e^{inx}$ of a function $f \in L^p[0, 2\pi]$, $p \geq 1$. We ask what conditions the sequence of numbers $(\lambda_n)_{n\in\mathbb{Z}}$ must satisfy that the series $\sum \lambda_n c_n e^{inx}$ is also the Fourier series of some function from $L^p[0, 2\pi]$. In other words, we consider the multiplier transformation T_λ defined by a sequence of numbers $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ by the formula

$$T_\lambda f \sim \sum \lambda_n c_n e^{inx} \quad \text{if} \quad f \sim \sum c_n e^{inx},$$

and ask under what assumptions on λ the operator T_λ is bounded in $L^p[0, 2\pi]$.

For $p = 2$ such a characterization is $\lambda \in l^\infty$, and for $p = 1$ the answer is also known (see [To86, p. 129]). The question for $1 < p < \infty$, $p \neq 2$ is much more difficult and still unsolved. A certain condition of the sequence of numbers $(\lambda_n)_{n\in\mathbb{Z}}$ which implies boundedness of the operator T_λ in $L^p[0, 2\pi]$ for $p > 1$ was given by Marcinkiewicz in the paper [M39f].

Theorem 17 (Marcinkiewicz's multiplier theorem 1939). *Let $1 < p < \infty$. If a sequence $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ is bounded and sums of differences over dyadic blocks are bounded, that is,*

$$\sup_n \sum_{2^n \leq |k| < 2^{n+1}} |\lambda_k - \lambda_{k-1}| \leq M < \infty, \tag{38}$$

then the operator T_λ is bounded in $L^p[0, 2\pi]$ and $\|T_\lambda f\|_p \leq C(\sup_{n\in\mathbb{Z}} |\lambda_n| + M)\|f\|_p$ for $f \in L^p$.

We can define the space of multipliers M_p for $1 \le p \le \infty$, as the space of all sequences $(\lambda_n)_{n\in\mathbb{Z}}$ such that $\|\sum \lambda_k c_k(f)e^{ikx}\|_p \le C\|f\|_p$ for any trigonometric polynomial f with a constant $C > 0$ independent of f. The infimum over all such C defines a norm and the space M_p becomes a Banach space. Figà–Talamanca (1965) additionally proved that this is a dual space for $1 < p \le 2$ and was even able to find its predual.

The following statements are true: $M_2 = l^\infty$, $M_p = M_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $M_p \subset M_q \subset l^\infty$ if $1 \le p \le q \le 2$. Theorem 16 of Marcinkiewicz means that if a sequence λ is bounded and satisfies (38), then it belongs to M_p.

Much more important, because of applications, is the corresponding Marcinkiewicz theorem for multiple series. For simplicity, let us write it in the two-dimensional case. Consider the multiplier transformation T_λ given by a double sequence $\lambda = (\lambda_{nm})_{n,m\in\mathbb{N}}$ with the formula

$$T_\lambda f \sim \sum \lambda_{nm} c_{nm} e^{i(nx+my)} \quad \text{as far as } f \sim \sum c_{nm} e^{i(nx+my)}$$

and dyadic intervals $I_k = \{i \in \mathbb{Z} : 2^{k-1} \le |i| < 2^k\}$, $J_l = \{j \in \mathbb{Z} : 2^{l-1} \le |j| < 2^l\}$, and let

$$\Delta_1\lambda_{nm} = \lambda_{n+1,m} - \lambda_{n,m}, \quad \Delta_2\lambda_{nm} = \lambda_{n,m+1} - \lambda_{n,m} \text{ and } \Delta_{1,2} = \Delta_1 \cdot \Delta_2.$$

THEOREM 17′ (Marcinkiewicz's multiplier theorem 1939). *Let $1 < p < \infty$. If for a double sequence $\lambda = (\lambda_{nm})_{n,m\in\mathbb{Z}}$ the following suprema are finite*:

$$A = \sup_{n,m} |\lambda_{n,m}|, \quad B_1 = \sup_{k,m} \sum_{n\in I_k} |\Delta_1\lambda_{n,m}|, \quad B_2 = \sup_{n,l} \sum_{m\in J_l} |\Delta_2\lambda_{n,m}|,$$
$$B_{1,2} = \sup_{k,l} \sum_{n\in I_k} \sum_{m\in J_l} |\Delta_{1,2}\lambda_{n,m}|,$$

then the operator T_λ is bounded in $L^p([0,2\pi]^2)$ and $\|T_\lambda f\|_p \le C(A + B_1 + B_2 + B_{1,2})\|f\|_p$ for $f \in L^p$.

As concrete examples of multipliers (λ_{nm}) Marcinkiewicz presented the following ones ([M39f], Thm 3):

$$\frac{m^2}{n^2+m^2}, \quad \frac{n^2}{n^2+m^2}, \quad \frac{|mn|}{n^2+m^2},$$

and informed that in this way some problem posed by Schauder is solved.

The corresponding multiplier theorem can be formulated also for the Fourier transform $\mathcal{F}$ in $L^p(\mathbb{R}^n)$. For a bounded measurable function m on $\mathbb{R}^n$ we define an operator T_m as follows:

$$T_m f(x) = \mathcal{F}^{-1}[m(\cdot)\mathcal{F}f(\cdot)](x), \; f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{39}$$

The function m is said to be an *L^p-multiplier* if

$$\|T_m f\|_p \le C_p\|f\|_p, \quad \text{for all } f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

In this case $T_m(\cdot)$ can be extended to $L^p(\mathbb{R}^n)$. The smallest constant C_p is the norm of this operator in $L^p(\mathbb{R}^n)$ and it is denoted by the symbol $\|T_m\|_p$. Note that $\|T_m\|_2 = \|m\|_\infty$ and if m is an L^p-multiplier, $1 < p < \infty$, then it also is an $L^{p'}$-multiplier and $\|T_m\|_{p'} = \|T_m\|_p$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

The Marcinkiewicz's multiplier theorem for the Fourier transform has the following form (let us formulate, for simplicity, only the one-dimensional case):

THEOREM 17″ (Marcinkiewicz's multiplier theorem 1939). *Let $m : \mathbb{R} \to \mathbb{R}$ be a bounded function of class C^1 on each dyadic set $(-2^{k+1}, -2^k) \cup (2^k, 2^{k+1})$ for $k \in \mathbb{Z}$. Assume that the derivative m' of the function m satisfies the condition*

$$\sup_{k\in\mathbb{Z}}\Big(\int_{-2^{k+1}}^{-2^k} |m'(t)|\,dt + \int_{2^k}^{2^{k+1}} |m'(t)|\,dt\Big) \le A < \infty. \tag{40}$$

Then m is an L^p-multiplier for all $1 < p < \infty$ and

$$\|T_m\|_p \le C \max\Big(p, \frac{1}{p-1}\Big)^6 (\|m\|_\infty + A).$$

The next known result on multipliers for Fourier integrals is the Hörmander–Mikhlin theorem. In 1956 Mikhlin [Mi56] proved Marcinkiewicz's result on Fourier integrals, and in 1960 Hörmander, in the paper [Ho60], gave a further generalization and simplification of the proofs. This result is sometimes called Hörmander–Mikhlin multiplier theorem, which in the simplest one-dimensional case has form: *Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded function on $\mathbb{R} \setminus \{0\}$ and satisfying either the Mikhlin condition $|xm'(x)| \le A$ or weaker the Hörmander condition*

$$\sup_{R>0} R \int_{R<|x|<2R} |m'(x)|^2\,dx \le A^2 < \infty. \tag{41}$$

Then m is an L^p-multiplier for all $p \in (1, \infty)$ and

$$\|T_m\|_p \le C \max\Big(p, \frac{1}{p-1}\Big)(\|m\|_\infty + A).$$

Moreover, T_m is of weak type $(1, 1)$.

Observe that in the one-dimensional case Theorem 17″ is stronger than the Hörmander–Mikhlin theorem, i.e., from the condition (41) follows the condition (40). But if we write these statements in higher dimensions, then the criteria of being multiplier in the Marcinkiewicz's theorem and the Hörmander–Mikhlin theorem are not comparable (see [Gr08], pp. 361–370). In addition, the assumption in the Marcinkiewicz theorem does not guarantee the weak type $(1, 1)$ of the mapping T_m (see Kislyakov [Ki88], p. 161).

Problems of Fourier multipliers with an extensive literature can be found in the books listed below, and further generalizations of the Marcinkiewicz and Hörmander–Mikhlin theorems, either weakening assumptions about λ or m or considering multipliers from spaces L^p to L^q for $1 \le p \le q \le \infty$, and also investigating multipliers for functions with values in Banach spaces, can be found in many works. Below some of them are cited.

The Fourier transform can be consider on groups. Let G be a compact Abelian group. A subset E of its dual group is called a *Marcinkiewicz set* if the multiplier $m = \chi_E$ is of weak type $(1, 1)$. The name "Marcinkiewicz set" as well as "quasi-Marcinkiewicz set" (and also a class of *Marcinkiewicz systems* and *quasi-Marcinkiewicz systems*) was introduced and studied by Kwapień and Pełczyński [KP80], and further information regarding these sets, together with examples, can be found in the paper by Kislyakov [Ki01].

We note further that Marcinkiewicz in a joint paper with Kaczmarz [MK38] gave conditions under which the sequence of numbers $\lambda = (\lambda_n)$ is an L^p-L^q-multiplier, and also that the expansions are with respect to any bounded orthonormal system on $[0,1]$ which is complete in $L^1[0,1]$.

[St70] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton NJ, 1970 [IV. 6. The Marcinkiewicz multiplier theorem, pp. 108–112].

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[EG77] R. E. Edwards, G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Ergeb. Math. Grenzgeb. 90, Springer, Berlin–New York, 1977 [1.1.4. The weak Marcinkiewicz multiplier theorem, pp. 5–17; Chapter 8. Strong forms of Marcinkiewicz multiplier theorem and Littlewood–Paley theorem for $\mathbb{R},\mathbb{T}$ and $\mathbb{Z}$, pp. 148–179].

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[To86] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Pure Appl. Math. 123, Academic Press, Orlando, 1986 [Marcinkiewicz multiplier theorem, pp. 326–327].

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4.3.4. *Convergence of Riemann sums and Marcinkiewicz–Salem conjecture* (1940). Marcinkiewicz, while staying in Paris, has written together with Raphaël Salem the paper [MS40], published in 1940, concerning Riemann sums.

Let $\mathbb{T} = [0,1) = \mathbb{R}/\mathbb{Z}$ with normalized Lebesgue measure m. For a measurable function f on $\mathbb{T}$ and $n \in \mathbb{N}$ we define the n-th Riemann sum of f as

$$R_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\Big(x + \frac{k}{n}\Big), \quad x \in \mathbb{T}. \tag{42}$$

In particular, when $x = 0$, we have the usual Riemann sums

$$R_n f = \frac{1}{n} \sum_{k=0}^{n-1} f\Big(\frac{k}{n}\Big).$$

– If f is Riemann integrable on $\mathbb{T}$, then for any $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} R_n f(x) = \int_0^1 f(t)\, dm. \tag{43}$$

– If f is only Lebesgue integrable on $\mathbb{T}$, then

$$\lim_{n\to\infty} \Big\| R_n f(\cdot) - \int_0^1 f(t)\, dm \Big\|_{L^1(\mathbb{T})} = 0. \tag{44}$$

Then it is natural to pose the question on pointwise convergence of these sums. The first investigations were done by Hahn (1914) who wanted to approximate Lebesgue integrals by the Riemann integrals. However, the first result is due to Jessen (1934):

– *if $f \in L^1(\mathbb{T})$ and if (n_k) is an increasing sequence of natural numbers in which next term divides the previous one, then*

$$\lim_{k\to\infty} R_{n_k} f(x) = \int_0^1 f(t)\, dm \quad \textit{for almost all } x.$$

Remark 2 (Marcinkiewicz–Salem 1940). The Jessen result is in some sense the best one, i.e., for the sequence $(2^n)_{n\geq 1}$ and for any positive increasing function w satisfying $\lim_{x\to\infty}\frac{w(x)}{\ln x}=0$ we can find a function f such that

$$\int_{\mathbb{T}} |f|\,w(|f|)\,dm<\infty \quad\text{and}\quad \int_{\mathbb{T}} \sup_{k\geq 0}|R_{2^k}f|\,dm=+\infty.$$

- Ursell (1937) and Marcinkiewicz–Salem (1940) showed the existence of a function $f\in L^1(\mathbb{T})$ such that $\limsup_{n\to\infty}|R_nf(x)|=+\infty$ for any x.
- Rudin (1964) showed even more, i.e., the existence of a function $f\in L^1(\mathbb{T})$ such that $\limsup_{n\to\infty}R_{2n+1}f(x)=+\infty$.

From this it follows that we cannot have convergence almost everywhere even for bounded functions.

Theorem 18 (Marcinkiewicz–Salem 1940).

(a) *If*

$$\int_{\mathbb{T}}[f(x+t)-f(x)]^2\,dx=O(t^{\varepsilon}),\quad \varepsilon>0, \tag{45}$$

then the sequence $(R_nf)_{n\geq 1}$ *is convergent almost everywhere to* $\int_{\mathbb{T}} f\,dm$.

(b) *If*

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{[f(x+t)-f(x)]^2}{t\,|\ln(t/2)|}\,dt\,dx<\infty, \tag{46}$$

then the sequence of averages $\big(A_nf=\frac{1}{n}\sum_{k=1}^{n}R_kf\big)f_{n\geq 1}$ *is convergent almost everywhere to* $\int_{\mathbb{T}} f\,dm$.

(c) *If*

$$\int_{\mathbb{T}}|f(x+t)-f(x)|\,dx=O\Big(\frac{1}{|\ln t|^p}\Big),\quad p>1, \tag{47}$$

then the sequence of averages $(A_nf)_{n\geq 1}$ *is convergent almost everywhere to* $\int_{\mathbb{T}} f\,dm$.

Note that condition (46) holds when for example

$$\int_{\mathbb{T}}[f(x+t)-f(x)]^2\,dx=O\Big(\frac{1}{\ln^2|\ln t|}\Big),$$

which is essentially weaker than (45). Moreover, if f is non-decreasing and $\int_{\mathbb{T}}|f(x)|^q\,dx$ is finite for some $q>1$, then (47) holds.

It is time to formulate a famous conjecture, namely the following one:

Marcinkiewicz–Salem conjecture (1940). *If* $f\in L^2(\mathbb{T})$, *then the sequence of averages* $(A_nf=\frac{1}{n}\sum_{k=1}^{n}R_kf)_{n\geq 1}$ *is convergent almost everywhere.*

Let us mention the result of Bourgain (1990) connected with this hypothesis: *If* $f\in L^2(\mathbb{T})$, *then the sequence* (R_nf) *has logarithmic density, i.e.,*

$$\frac{1}{\ln N}\sum_{n=1}^{N}\frac{1}{n}R_nf\to\int_{\mathbb{T}} f\,dm\quad a.e.$$

More information, proofs, connection with number theory and generalizations related to convergence almost everywhere can be found in the Ruch–Weber paper (2006) and other papers mentioned below:

[Bo90] J. Bourgain, *Problems of almost everywhere convergence related to harmonic analysis and number theory*, Israel J. Math. 71 (1990), 97–127.

[Je34] B. Jessen, *On the approximation of Lebesgue integrals by Riemann sums*, Ann. of Math. (2) 35 (1934), 248–251.

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[Ru64] W. Rudin, *An arithmetic property of Riemann sums*, Proc. Amer. Math. Soc. 15 (1964), 321–324.

[We04] M. Weber, *A theorem related to Marcinkiewicz–Salem conjecture*, Results Math. 45 (2004), 169–184.

[We05] M. Weber, *Almost sure convergence and square functions of averages of Riemann sums*, Results Math. 47 (2005), 340–354.

4.3.5. *Marcinkiewicz's theorem on universal primitive functions* (1935). Luzin (1915) showed that every Lebesgue measurable and almost every finite function on an interval $[a, b]$ is almost everywhere derivative of a continuous function (a proof can be found in the Saks book [Sa37], pp. 217–218). Marcinkiewicz (1935) generalized this theorem in [M35a] by proving the following remarkable fact: there is a continuous function F, which is a generalized primitive function (antiderivative) for every a.e. finite Lebesgue measurable function f, that is, F is a universal generalized primitive function. Marcinkiewicz showed also that most functions are universal primitive functions, since in the class of continuous functions the functions which are not universal generalized primitive functions form a set of the first category in $C[a, b]$. Marcinkiewicz not only proved the existence of universal primitive function, but he also the first to use the word "universal" in such context and the first to show that a set of universal elements is residual.

Theorem 19 (Marcinkiewicz's theorem on universal primitive functions 1935). *Let $[a, b] \subset \mathbb{R}$ and let $(h_n)_{n=1}^{\infty}$ be a fixed sequence of nonzero real numbers converging to zero. Then there exists a continuous function $F : [a, b] \to \mathbb{R}$ having the following property: if $f : [a, b] \to \mathbb{R}$ is any Lebesgue measurable function, then there is a subsequence (h_{n_k}) of $(h_n)_{n=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} = f(x) \quad \textit{almost everywhere on } [a, b].$$

Such functions F constitute a residual set in $C[a, b]$.

Note that one and the same function F works for all functions f. Of course the subsequence depends on f. The function F may be called *generalized primitive function* (*antiderivative*) of f with respect to the given sequence $(h_n)_{n=1}^{\infty}$ and it is clear that such an F may be a generalized primitive function of many functions not equivalent to f.

Marcinkiewicz's Theorem 19 with proof can be found in the books by Saks (1937), Bruckner (1978), Stromberg (1981) and Wise-Hall [WH93]:

[Sa37] S. Saks, *Theory of the Integral*, 2nd rev. ed., Monografie Matematyczne VII, Warszawa-Lwów, 1937 [Marcinkiewicz theorem, p. 218].

[Br78] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math. 659, Springer, Berlin-New York, 1978 [Theorem 3.3. Marcinkiewicz, pp. 82–83].

[St81] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth International, Belmont, 1981 [Marcinkiewicz theorem with the proof, pp. 316–317].

[WH93] G. L. Wise, E. B. Hall, *Counterexamples in Probability and Real Analysis*, Oxford Univ. Press, New York, 1993 [Marcinkiewicz theorem with the proof in Example 3.15, pp. 78–79].

Several authors have obtained strengthenings, generalizations and variants of Marcinkiewicz Theorem 19. Hoàng Tụy [HT59], [HT60] shows theorem for essential left and right derivative numbers of f at almost each x, a martingale version is due to Lamb [La74], Aversa and Carrese [AC83] obtain an n-dimensional version for interval functions, Grande [Gr84] gives a Banach space-valued generalization, that is, for functions $f : [0,1] \to X$, where X is a Banach space. Cater [Ca89] replaces the difference quotient $(F(x+h) - F(x))/h$ by certain higher-order difference quotients, and Gan and Stromberg [GS94] obtain the generalization of Marcinkiewicz's theorem for functions $f : [0,1]^n \to \mathbb{R}^n$. Smooth universal Marcinkiewicz functions were constructed by Krotov in [Kr91].

Joó [Jo89] studied the problem when one replaces a.e. convergence by convergence in $L^p[0,1]$ for any $0 < p < 1$. He (also Herzog and Lemmert [HL06]) showed the existence of a universal primitive F in the space $C[0,1]$ such that to each function $f \in L^p[0,1]$ there is a subsequence of $(F(x+\lambda_n) - F(x))/\lambda_n$ with limit f in $L^p[0,1]$. Several authors showed that one cannot choose here $p \geq 1$ (see Bogmér–Sövegjártó [BS87], Buczolich [Bu87] and Horváth [Ho87]). Herzog and Lemmert [HL09] proved a universality theorem from which we can deduce that F may be chosen to be Hölder continuous for each exponent $\alpha \in (0,1)$. Of course there are no Lipschitz continuous universal primitives since each Lipschitz continuous function is differentiable almost everywhere.

The general definition of the universality was given by Grosse-Erdmann in 1999. He presented this concept with several examples and references in a survey article [GE99]. Some related problems are raised in Laurinčikas [La03].

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[BL66] A. M. Bruckner, J. L. Leonard, *Derivatives*, Amer. Math. Monthly 73 (1966), no. 4, Part II, 24–56 [Marcinkiewicz result, p. 29].

[Bu87] Z. Buczolich, *On universal functions and series*, Acta Math. Hungar. 49 (1987), 403–414.

[Ca89] F. S. Cater, *Some higher-dimensional Marcinkiewicz theorems*, Real Anal. Exchange 15 (1989/90), 269–274.

[GS94] X.-X. Gan, K. R. Stromberg, *On universal primitive functions*, Proc. Amer. Math. Soc. 121 (1994), 151–161.

[Gr84] E. Grande, *Sur un théorème de Marcinkiewicz*, Problemy Mat. 4 (1984), 35–41.

[GE99] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345–381 [Theorem 12 (Marcinkiewicz), p. 362].

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[Ho87] M. Horváth, *On multidimensional universal functions*, Studia Sci. Math. Hungar. 22 (1987), 75–78.

[Jo89] I. Joó, *On the divergence of eigenfunction expansions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 32 (1989), 3–36.

[Kr91] V. G. Krotov, *On the smoothness of universal Marcinkiewicz functions and universal trigonometric series*, Izv. Vyssh. Uchebn. Zaved. Mat. 1991, no. 8, 26–31; English transl.: Soviet Math. (Iz. VUZ) 35 (1991), no. 8, 24–28.

[La74] C. W. Lamb, *Representation of functions as limits of martingales*, Trans. Amer. Math. Soc. 188 (1974), 395–405.

[La03] A. Laurinčikas, *The universality of zeta-functions*, Acta Appl. Math. 78 (2003), 251–271.

4.3.6. *Marcinkiewicz theorem on Perron integral and Marcinkiewicz–Zygmund integral.* If a function $f:[a,b]\to\mathbb{R}$ has derivative f' which is Riemann integrable, then from the fundamental theorem of calculus we have the equality

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{48}$$

The equality (48) is not always true, even if f is differentiable on $[a,b]$, since the derivative can be unbounded and thus not Riemann integrable on $[a,b]$. We would like to have such an integral, which has sense for all derivatives and which ensure the equality (48). For example for the function $f(x) = x^2\cos\frac{\pi}{x^2}$ for $0 < x \le 1$ and $f(0) = 0$ it yields

$$\int_{I_n} |f'(x)|\,dx = \frac{1}{2n}, \text{ where } I_n = \Big[\Big(\frac{2}{4n+1}\Big)^{1/2}, \frac{1}{(2n)^{1/2}}\Big]$$

and, hence,

$$\int_0^1 |f'(x)|\,dx = \infty.$$

Thus f' is not Lebesgue integrable on $[0,1]$, nevertheless the improper integral

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 f'(x)\,dx$$

exists. Consequently, the Lebesgue integral does not solve the problem.

Denjoy (1912) introduced an integral having the required property. The narrow Denjoy integral of function f is defined by existence of a continuous indefinite integral F on $[a,b]$ such that $F' = f$ almost everywhere in $[a,b]$ with some technical conditions. This integral is equivalent with the Perron integral defined in 1914 with the help of major and minor function of f. For the Perron integral the formula (48) is true if the function f is differentiable on $[a,b]$.

Let $f:[a,b]\to\mathbb{R}$. Then function $M:[a,b]\to\mathbb{R}$ is called a *major* function of f if $M(a) = 0$ and $\underline{D}M(x) \ge f(x)$ for $x\in[a,b]$, and $\underline{D}M(x) = \liminf_{h\to 0}\frac{M(x+h)-M(x)}{h}$, and function $m : [a,b] \to \mathbb{R}$ is called a *minor* function of f if $m(a) = 0$ and $\overline{D}m(x) \le f(x)$ for $x \in [a,b]$, where $\overline{D}m(x) = \limsup_{h\to 0} \frac{m(x+h)-m(x)}{h}$.

A function $f : [a,b] \to \mathbb{R}$ is said to be *integrable in the sense of Perron* (P-integrable) on $[a,b]$ if there exist major function M of f and minor function m of f, and $\inf_M M(b) = \sup_m m(b) = I$. Their common value I is called the Perron integral of f on $[a,b]$ and is denoted by (P) $\int_a^b f(x)\,dx$. If f is P-integrable on $[a,b]$, then $F(u) = (\mathrm{P})\int_a^u f(x)\,dx$, $u \in [a,b]$, is continuous and $F' = f$ almost everywhere on $[a,b]$, and hence we are getting that f is Lebesgue measurable (F need not be absolutely continuous). Moreover, if f is P-integrable on $[a,b]$ and nonnegative, then f is also Lebesgue integrable on $[a,b]$.

One of the unexpected results on the Perron integral is Marcinkiewicz's theorem contained in the book by Saks ([Sa37], p. 253) and not published in any paper of Marcinkiewicz (cf. Bullen [Bu90], p. 12).

THEOREM 20 (Marcinkiewicz theorem on Perron integral 1937). *A measurable function $f : [a,b] \to \mathbb{R}$ is Perron integrable if and only if it has one continuous major and one continuous minor function.*

This theorem was proved also by Tolstov [To39] and Denjoy [De49]. Generalizations of Theorem 20 on other Perron type integrals and on functions having possibly infinite values were formulated by McShane [Mc42], Frenkel and Cotlar [FC50], Sarkhel [Sa78]. Marcinkiewicz's theorem is true for AP-integral (approximately continuous Perron integral), CP-integral (Cesàro–Perron integral) – see Bullen [Bu90]. Failure of the Marcinkiewicz theorem for SCP-integral (symmetric Cesàro–Perron integral) was proved by Sklyarenko [Sk99], for integrals defined by symmetric derivatives was proved by Skvortsov and Thomson [ST96] (see Thomson [Th94]), and for P_d-integral (dyadic Perron integral) was noticed by Skvortsov [Sk96]. Research, whether for a given integral Marcinkiewicz's theorem is true or not, continues to this day.

Marcinkiewicz's theorem on existence of the Perron integral appeared in differential equations (in Peano theorem) – see e.g. Bullen and Výborný [BV91].

In trigonometric series the fundamental problem was to define an integral in such a way that if a trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is convergent everywhere to a function $f(x)$, then $f(x)$ is integrable and the coefficients a_n and b_n are given by usual Fourier formulas. This problem was solved in many ways starting form Denjoy (1916).

Marcinkiewicz and Zygmund [MZ36] also gave such a way by defining the inverse of Borel derivative and using the Perron method. Now, a function M on $[a,b]$ is major of $f : [a,b] \to \mathbb{R}$, if $M(a) = 0$ and

$$\underline{B}_s M(x) = \liminf_{h\to 0} \frac{1}{h}\int_0^h \frac{M(x+t)-M(x-t)}{2t}\,dt \ge f(x),$$

and m on $[a,b]$ is minor of f, if $m(a) = 0$ and

$$\overline{B}_s m(x) = \limsup_{h\to 0} \frac{1}{h}\int_0^h \frac{m(x+t)-m(x-t)}{2t}\,dt \le f(x).$$

Marcinkiewicz–Zygmund integral (MZ-integral) is by definition

$$I = (\mathrm{MZ})\int_a^b f(x)\,dx := \inf_M M(b) = \sup_m m(b).$$

This integral has the required fundamental property: if the trigonometric series is convergent everywhere to $f(x)$, then f is integrable in the sense of Marcinkiewicz–Zygmund on $[0, 2\pi]$.

[Sa37] S. Saks, *Theory of the Integral*, 2nd rev. ed., Monografie Matematyczne VII, Warszawa-Lwów, 1937 [Marcinkiewicz theorem, p. 253].

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[Sk96] V. A. Skvortsov, *On the Marcinkiewicz theorem for the binary Perron integral*, Mat. Zametki 59 (1996), 267–277; English transl.: Math. Notes 59 (1996), 189–195.

[ST96] V. A. Skvortsov, B. S. Thomson, *Symmetric integrals do not have the Marcinkiewicz property*, Real Anal. Exchange 21 (1995/96), 510–520.

[To39] G. P. Tolstoff, *Sur l'intégrale de Perron*, Mat. Sb. 5 (1939), 647–660.

[VS71] I. A. Vinogradova, V. A. Skvortsov, *Generalized integrals and Fourier series*, in: Mathematical Analysis 1970, Itogi Nauki, Akad. Nauk SSSR VINITI, Moscow, 1971, 67–107; English transl.: J. Soviet Math. 1 (1973), 677–703.

4.4. Fourier series and orthogonal series. Consider a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{49}$$

or, formally equivalent, series in the complex form

$$\sum_{n=-\infty}^{\infty} c_n\, e^{inx}, \tag{50}$$

assuming $b_0 = 0$ we get $c_n = (a_n - ib_n)/2$, $c_{-n} = (a_n + ib_n)/2$.

A trigonometric series (49) is a Fourier series of a 2π-periodic function $f \in L^1[-\pi,\pi]$, if the coefficients (Fourier coefficients) are given by the formulas

$$\begin{gathered} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \\ c_n = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \end{gathered} \tag{51}$$

The partial sums $S_n f$, $n \geq 1$, of a Fourier series of a function f are

$$S_n f(x) = \sum_{|k|\leq n} \hat{f}(n)e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-y)f(y)\,dy, \tag{52}$$

where D_n is the *Dirichlet kernel* of the form

$$D_n(x) = \sum_{|k|\leq n} e^{ikx} = \frac{\sin(n+1/2)x}{\sin(x/2)}, \quad n = 0,1,\ldots,$$

with value $2n+1$ at $x = 0 \pmod{2\pi}$. The Cesàro means $\sigma_n f$ are

$$\sigma_n f(x) = \frac{S_0 f(x) + S_1 f(x) + \ldots + S_n f(x)}{n+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x-y)f(y)\,dy,$$

where F_n denotes the *Fejér kernel*

$$F_n(x) = \frac{D_0 f(x) + D_1 f(x) + \ldots + D_n f(x)}{n+1} = \frac{1}{n+1}\,\frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{x}{2}}.$$

For $f \in L^p$ with $1 < p < \infty$ it yields that $\lim_{n\to\infty}\|f - S_n f\|_p = \lim_{n\to\infty}\|f - \sigma_n f\|_p = 0$ and for $f \in L^1$ we have $\lim_{n\to\infty}\|f - \sigma_n f\|_1 = 0$. On the other hand, if $f \in L^1$, then the Fourier sums can be divergent almost everywhere (Kolmogorov 1923) and the problem of almost everywhere convergence of Fourier series became an important object of research of many mathematicians around the world, including Marcinkiewicz and Zygmund.

4.4.1. *Pointwise convergence of Fourier series.* In this part we investigate 2π-periodic functions, therefore functions or convergence of functions will be considered either on $[-\pi,\pi]$ or $[0,2\pi]$.

The first paper of Marcinkiewicz [M33] contains a short proof of Kolmogorov's theorem (1924) on convergence of partial sums of lacunary Fourier series: if $f \in L^2$ and $\lambda_{n+1}/\lambda_n > q > 1$, then $S_{\lambda_n} f$ is convergent almost everywhere to f. A new proof deserved attention because of its brevity and clarity.

In the second paper, Marcinkiewicz [M34] generalized results of Wiener (1924) on the functions of finite p-variation with $p > 0$ (Wiener considered only the case $p = 2$). Recall that the function $f : [a,b] \to \mathbb{R}$ has finite *p-variation*, if

$$V_p(f) = V_p(f;a,b) = \Big(\sup_{\Pi}\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p\Big)^{1/p} < \infty,$$

where supremum is taken over all partitions $\Pi : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a,b]$. The collection of all such functions of finite p-variation is denoted by $V_p[a,b]$, and by $V_p[0,2\pi]$ we denote 2π-periodic functions having finite p-variation. Marcinkiewicz showed that ([M34], Theorem 1 and 2) if $f \in V_p[a,b]$ $(0 < p < \infty)$, then the function

$$\varphi_p(x) = \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|^{1/p}}$$

is finite almost everywhere and $\varphi_p \in L^p[a,b]$ with $\|\varphi_p\|_p \le V_p[a,b]$. He also proved that if $f \in V_p[0,2\pi]$ $(0 < p < \infty)$, then

$$\omega(\delta; f)_p := \sup_{|h|\le\delta} \Big(\int_0^{2\pi} |f(x+h) - f(x)|^p\,dx\Big)^{1/p} \le V_p(f; 0, 3\pi)\,\delta^{1/p} \text{ for } 0 < \delta \le \pi. \quad (53)$$

From this estimate we obtain the following results of Marcinkiewicz: *If* $f \in V_p[0,2\pi]$ $(p \ge 1)$*, then the coefficients of the Fourier series of a function* f *are of order* $O(n^{-1/p})$ *and the Fourier series is convergent almost everywhere to* f.

The estimate (53) has appeared in textbooks and we can find it, for example, in [Ta79], pp. 17–18.

In the next paper [M35d] from 1935 Marcinkiewicz improved the result of Hardy–Littlewood (1932) on almost everywhere convergence of Fourier series.

Theorem 21 (Marcinkiewicz's test 1935).

(a) *If, for* x *belonging to the set* E *of positive measure, we have*

$$\frac{1}{t}\int_0^t |f(x+u) - f(x)|\,du = O\Big(\frac{1}{\log 1/|t|}\Big), \quad (54)$$

then $(S_n f)$ *is convergent almost everywhere in* E.

(b) *If* $f \in L^1[-\pi,\pi]$ *and*

$$\int_0^\pi \omega_1(f,t)\,\frac{dt}{t} < \infty, \quad \textit{where} \quad \omega_1(f,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|\,dx, \quad (55)$$

then $S_n f(x) \to f(x)$ *for almost all* $x \in [-\pi,\pi]$.

Marcinkiewicz also showed that the result in (a) is the best possible one, in the following sense (announced in [M35d] and proved in [M36a], Thm 2): if the function $\omega : \mathbb{R} \to (0,\infty)$ is even, nondecreasing in some interval $(0,\delta)$, $0 < \delta \le 1/3$ and such that $\lim_{t\to 0}\omega(t) = 0$, $\lim_{t\to 0}\omega(t)\log\frac{1}{|t|} = +\infty$, then there exists a function $f \in L^1$ for which

$$\frac{1}{t}\int_0^t |f(x+u) - f(x)|\,du = O[\omega(t)]$$

almost everywhere in $[0,2\pi]$, but the Fourier series is divergent almost everywhere.

Theorem 21(a) with proof can be found e.g. in the books by Zygmund ([Zy59], II, pp. 170–172), Alexits ([Al61], pp. 320–326) and Bary ([Ba64], I, pp. 417–421), where also the proof of optimality of this theorem appears ([Ba64], I, pp. 443–447). Moreover, Theorem 21(b) with proof can be found in the books written by Zygmund ([Zy59], II, p. 172), Torchinsky ([To86], p. 7), Bruckners and Thomson ([BBT97], p. 683).

Marcinkiewicz (1939) proved an interesting generalization of the Plessner theorem (1925), extending the case $p = 2$ to $1 \le p \le 2$ (for $p = 1$ we obtain the usual Dini test):

THEOREM 22 (Marcinkiewicz 1939). *Let $1 \le p \le 2$. If $f \in L^p$ and*

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(x+t) - f(x-t)|^p}{t} \, dt \, dx < \infty, \tag{56}$$

then the Fourier series of the function f is convergent almost everywhere.

Theorem 22, together with the proof, can be found e.g. in the book by Bary [Ba64], I, pp. 379–380.

The year 1966 was a breakthrough for research on convergence almost everywhere of Fourier series. Then the Swedish mathematician Lennart Carleson [Car66] proved the Luzin conjecture from 1913 that if $f \in L^2(-\pi, \pi)$, then the Fourier series is convergent almost everywhere. Equivalent formulation is: if $(c_n)_{n=-\infty}^{\infty} \in l^2$ is a sequence of complex numbers, then the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is convergent almost everywhere. Hence, an analogue for Fourier integrals can be deduced: if $f \in L^2(\mathbb{R})$, then the functions $g_\alpha(x) = \int_{-\alpha}^{\alpha} f(t) e^{itx} \, dt$ are convergent almost everywhere, when $\alpha \to \infty$. In 1967, Richard Hunt, extended the result of Carleson to the spaces $L^p(-\pi, \pi)$ for $1 < p < \infty$, and since then the assertion is called the Carleson–Hunt theorem. A new elegant proof of this theorem was given by Charles Fefferman [Fe73] in 1973.

I guess that the theorem which Marcinkiewicz might most have liked to see is the Carleson–Hunt theorem.

In the twenties also papers on almost everywhere divergence of Fourier series appeared. In 1923 Kolmogorov constructed a function $f \in L^1$ such that the partial Fourier sums $S_n f$ are unbounded almost everywhere, and so divergent almost everywhere. He also added an example of a function $f \in L^1$ which Fourier sums $S_{2^n} f$ are divergent almost everywhere. Three years later, Kolmogorov noticed the existence of a function $f \in L^1$ with divergent Fourier sums at every point.

In 1927 Kolmogorov together with Menshov published the paper [KM27] in which they informed, without proof, about the existence of a function $f \in L^2$ whose Fourier series after a permutation of terms is divergent almost everywhere. It was not possible to reproduce their proof despite requests addressed even to Kolmogorov. Thus, there was not known neither a function nor permutation. A short method of construction was found only in 1960 by Z. Zahorski [Za60] (see also [Ul83], p. 71).

In 1936 Marcinkiewicz modified the construction of Kolmogorov from 1923 in his paper [M36a]. This necessary modification was not at all obvious.

THEOREM 23 (Marcinkiewicz's example 1936). *There exists a function $f \in L^1[0, 2\pi]$ such that the Fourier sums $S_n f$ are divergent almost everywhere on $[0, 2\pi]$ and*

$$\limsup_{n \to \infty} |S_n f(x)| < \infty$$

for almost all $x \in [0, 2\pi]$, that is, the Fourier series of function f is boundedly divergent almost everywhere on $[0, 2\pi]$.

The Kolmogorov and Marcinkiewicz constructions can be found in the Zygmund book [Zy59], pp. 305–308 and 308–310, in the Bary book [Ba64], pp. 430–442, and in review articles of Ul'yanov [Ul57], pp. 95–99 and 102–106, [Ul83], pp. 57–65.

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[Ul57] P. L. Ul'yanov, *On the divergence of Fourier series*, Uspekhi Mat. Nauk (N.S.) 12 (1957), no. 3, 75–132 (Russian) [Marcinkiewicz example, pp. 102–106, Marcinkiewicz test, pp. 109–111, and its sharpness, pp. 111–116].

[Ul83] P. L. Ul'yanov, *A. N. Kolmogorov and divergent Fourier series*, Uspekhi Mat. Nauk 38 (1983), no. 4, 51–90 [Marcinkiewicz example, p. 65, Marcinkiewicz test, p. 68]; English transl.: Russian Math. Surveys 38 (1983), no. 4, 57–100.

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[Zh87] H. S. Zhao, *The Marcinkiewicz theorem for Fourier series on compact Lie groups*, Chinese Ann. Math. Ser. A 8 (1987), 693–702; English summary: Chinese Ann. Math. Ser. B 9 (1988), 148–149.

4.4.2. *Orthogonal series.* A sequence $\Phi = (\varphi_n)_{n=1}^{\infty}$, where $\varphi_n : [a,b] \to \mathbb{R}$ $(n = 1, 2, \dots)$ is called an *orthogonal system* in $L^2[a,b]$, if

$$\int_a^b \varphi_m(t)\,\varphi_n(t)\,dt = 0 \quad \text{for } m \neq n \quad \text{and} \int_a^b \varphi_n^2(t)\,dt = \lambda_n > 0 \quad (m, n = 1, 2, \dots).$$

If, in addition, $\lambda_1 = \lambda_2 = \ldots = 1$, then the system is called *orthonormal*.

We will discuss three results of Marcinkiewicz from the general theory of orthogonal series: the problem of almost everywhere convergence of a subsequence of any orthonormal system (1936), a generalization of the Hausdorff–Young inequality (1937) and the theorem of Marcinkiewicz on Haar system (1937).

In 1936 Marcinkiewicz proved an interesting theorem for general orthonormal systems ([M36c], Thm A).

THEOREM 24 (Marcinkiewicz 1936). *For any orthonormal system $\Phi = (\varphi_n)_{n=1}^{\infty}$ of functions defined on $[0,1]$ there exists an increasing sequence $(N_k)_{k=1}^{\infty}$ of natural numbers such that for any series*

$$\sum_{n=1}^{\infty} a_n\varphi_n(x) \quad \textit{with} \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty, \tag{57}$$

the sequence $\big(S_{N_k}(x) = \sum_{n=1}^{N_k} a_n\varphi_n(x)\big)_{k=1}^{\infty}$ is almost everywhere convergent on $[0,1]$ and we have the estimate

$$\Big\|\sup_{m\geq 1} |S_{N_m}(x)|\Big\|_2 \leq C\Big(\sum_{n=1}^{\infty} |a_n|^2\Big)^{1/2},$$

with constant C independent of the system Φ.

A subsequence $(N_k)_{k=1}^{\infty}$ depends only on the system $\Phi = (\varphi_n)_{n=1}^{\infty}$. For different orthonormal systems the sequence (N_k) can be different. For example, for the Haar system we can take $N_k = k$ and for the trigonometric system $N_k = 2^k$.

If instead of $(a_n) \in l^2$ we have a stronger assumption, then we can find a universal sequence (N_k) good for all orthonormal systems, for example, if $\sum_{n=1}^{\infty} |a_n|^2 \ln n < \infty$ and $f \in L^2[0,1]$ is a sum of series $\sum_{n=1}^{\infty} a_n\varphi_n(x)$ in the $\|\cdot\|_2$-norm, then the sequence $S_{2^n}(x) = \sum_{k=1}^{2^n} a_k\varphi_k(x)$ is convergent almost everywhere to $f(x)$.

By using Theorem 24 Marcinkiewicz showed the following ([M36c], Thm B): *if $1 \leq p < \frac{6}{5}$, then there is an $f \in L^p[0,2\pi]$ and a rearrangement of the Fourier series of f such that the new obtained series $a_0/2 + \sum_{k=1}^{\infty}(a_{n_k}\cos n_k x + b_{n_k}\sin n_k x)$ diverges unboundedly almost everywhere in $[0,2\pi]$*. In 1957 Ul'yanov generalized this result to $1 \leq p < 2$ (see Bary [Ba64], I, pp. 480–482).

Marcinkiewicz's Theorem 24 is cited e.g. in the following books:

[KS58] S. Kaczmarz, H. Steinhaus, *Theory of Orthogonal Series*, Fizmatgiz, Moscow, 1958 (Russian).

[Si59] R. Sikorski, *Real Functions*, Vol. II, Monografie Matematyczne 37, PWN, Warszawa, 1959 (Polish) [Marcinkiewicz result, p. 148].

[Al61] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York, 1961 [2.10.2. Marcinkiewicz–Menchoff theorem, pp. 152–155 and 158].

[Ga70] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Lectures in Advanced Math. 4, Markham, Chicago, 1970 [Marcinkiewicz theorem, pp. 79–80].

[KS84] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (Russian) [Theorem 8.7. Marcinkiewicz theorem with the proof and estimate $C \leq 6\sqrt{3}$, pp. 317–320].

Let $\Phi = (\varphi_n)_{n=1}^{\infty}$ be an arbitrary orthogonal system in $L^2(a,b)$ and let the interval (a,b) be finite or infinite. If $\|\varphi_n\|_2 \leq C$ for any $n \in \mathbb{N}$, then considering the Fourier coefficients

$$c_n = \int_a^b f(x)\varphi_n(x)\,dx \tag{58}$$

of the function f with respect to the system Φ we must assume that $f \in L^2(a,b)$ to ensure the existence of the integrals (58). If $\|\varphi_n\|_\infty \le C$, then the integrals (58) exist for $f \in L^1(a,b)$. Marcinkiewicz and Zygmund [MZ37b] assumed that $\varphi_n \in L^r(a,b)$ for $2 \le r \le \infty$. Then integrals (58) exist for $f \in L^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

Marcinkiewicz and Zygmund [MZ37b] gave the following generalization of the Hausdorff–Young theorem (for trigonometric system and $r = \infty$) and the F. Riesz theorem (1923, for uniformly bounded orthonormal system and $r = \infty$).

THEOREM 25 (Marcinkiewicz and Zygmund 1937). *Let $\Phi = (\varphi_n)_{n=1}^\infty$ be an orthonormal system in $L^2(a,b)$ such that $\|\varphi_n\|_r = M_n < \infty$ for some $r \in (2,\infty]$ and arbitrary $n \in \mathbb{N}$. Assume also that p and q satisfy the equality $\frac{r'}{p} + \frac{2-r'}{q} = 1$.*

(a) *If $r' \le p \le 2$ and $f \in L^p(a,b)$, then $(\sum_{n=1}^\infty |c_n|^q\, M_n^{2-q})^{1/q} \le \|f\|_p$.*

(b) *If $2 \le p \le r$ and the sequence $a = (a_n)_{n=1}^\infty$ satisfies the conditions*

$$I_q(a) = \Big(\sum_{n=1}^\infty |a_n|^q\, M_n^{2-q}\Big)^{1/q} < \infty \quad \textit{and} \quad \sum_{n=1}^\infty |a_n|^2 < \infty,$$

then there exists $f \in L^p(a,b)$ such that $f = \lim_{n\to\infty} \sum_{k=1}^n a_k\varphi_n$ in $L^p(a,b)$ and $\|f\|_p \le I_q(a)$.

Note that if the numbers M_n are bounded from below by a positive constant, then the inequality $\sum_{n=1}^\infty |a_n|^q\, M_n^{2-q} < \infty$ implies $\sum_{n=1}^\infty |a_n|^q < \infty$ and so $\sum_{n=1}^\infty |a_n|^2 < \infty$ (the assumption $2 \le p \le r$ and the relation between p and q gives $1 \le q \le 2$). Hence, the condition of convergence $\sum_{n=1}^\infty |a_n|^2 < \infty$ may be omitted in the statement of Theorem 25. This is, in particular, the case when the interval (a,b) is finite, because

$$M_n = \|\varphi_n\|_r \ge (b-a)^{1/r-1/2}\|\varphi_n\|_2 = (b-a)^{1/r-1/2}.$$

Marcinkiewicz and Zygmund also proved a generalization of the Paley theorem (1931, $r = \infty$ and $M_1 = M_2 = \ldots = M$).

THEOREM 26 (Marcinkiewicz and Zygmund 1937). *Let $\Phi = (\varphi_n)_{n=1}^\infty$ be an orthonormal system in $L^2(a,b)$ such that $\|\varphi_n\|_r = M_n < \infty$ for some $r \in (2,\infty]$ and arbitrary $n \in \mathbb{N}$. Assume also that $M_1 \le M_2 \le \ldots M_n \le \ldots$.*

(a) *If $r' < p \le 2$ and $f \in L^p(a,b)$, then*

$$\Big(\sum_{n=1}^\infty |c_n|^p\, M_n^{r(p-2)/(r-2)}\, n^{(r-1)(p-2)/(r-2)}\Big)^{1/p} \le A(p,r)\|f\|_p.$$

(b) *If $2 \le p < r$ and the sequence $a = (a_n)_{n=1}^\infty$ satisfies the condition*

$$J_{r,p}(a) = \Big(\sum_{n=1}^\infty |a_n|^p\, M_n^{r(p-2)/(r-2)}\, n^{(r-1)(p-2)/(r-2)}\Big)^{1/p} < \infty,$$

then there exists $f \in L^p(a,b)$ such that $f = \lim_{n\to\infty} \sum_{k=1}^n a_k\varphi_n$ in $L^p(a,b)$ and $\|f\|_p \le B(p,r)J_{r,p}(a)$.

(c) *Moreover, $B(p,r) \le C\,\frac{r-2}{r-p}\,p$ for some $C > 0$ and $A(p,r) = B(p',r)$.*

Marcinkiewicz and Zygmund obtained the proof of Theorem 25 by using the interpolation theorem of M. Riesz. In [Zy56] Zygmund proved the original Paley theorem using the Marcinkiewicz interpolation theorem between strong type $(2,2)$ and weak type $(1,1)$ of the linear operator $Tf = (nc_n) = (n\int_a^b f(t)\varphi_n(t)\,dt)$. An application of the Marcinkiewicz interpolation theorem gives also some other generalizations of the Hausdorff–Young theorem (see Edwards [Ed82], pp. 192–197).

In [SW58] Stein and Weiss first proved a theorem on interpolation of operators with change of measures, as a generalization of the Riesz–Thorin and Marcinkiewicz theorems, and then used it to prove weighted version of the Paley theorem with $M_n^* = \max_{1\le k\le n} M_k$ instead of M_n with the increases monotonically assumption. Bullen [Bu61] considers, firstly, the cases of equality in the Hausdorff–Young theorems and then also observed that the assumption of increasing monotonicity of (M_n) can be a little weaker, namely that for some $a > 1$ and all $m, n \in \mathbb{N}$, $m < n$, we have $\max_{a^m+1\le k\le a^{m+1}} M_k \le C\max_{a^n+1\le k\le a^{n+1}} M_k$. Moreover, Kirillov [Ki98] proved that the monotone increasing assumption on the sequence (M_n) cannot be discarded.

In [Kol92] Kolyada gave a proof of the Marcinkiewicz–Zygmund Theorem 26(b) using elementary inequalities for numbers and also extended this theorem to Orlicz classes. Theorem 26 with new estimates in Lorentz spaces $L^{p,q}$ was given by Kirillov [Ki99].

The Hausdorff–Young and Riesz theorems were also considered for some concrete nonorthogonal systems but similar to orthogonal ones, as, for example, S. Verblunsky (1954) considered in $L^2[0,2\pi]$ the system $(\exp(i\chi_n x))_{n\in\mathbb{Z}}$, where $\chi_{-n} = -\chi_n$ and $0 = \chi_0 < \chi_1 < \ldots$ are positive solutions of the equation $x + h\tan\pi x = 0$ with $h > 0$ fixed. In [Ro01] Rodionov studied expansions of functions in the space L^p with respect to systems similar to orthogonal ones to include also the result by Verblunsky and proved Marcinkiewicz–Zygmund type theorems for such systems.

[Ed82] R. E. Edwards, *Fourier Series. Vol. 2. A Modern Introduction*, 2nd ed., Grad. Texts in Math. 85, Springer, New York–Berlin, 1982.

[Bu61] P. S. Bullen, *Properties of the coefficients of orthonormal sequences*, Canad. J. Math. 13 (1961), 305–315.

[Ki98] S. A. Kirillov, *On a theorem of Marcinkiewicz and Zygmund*, Mat. Zametki 63 (1998), 386–390; English transl.: Math. Notes 63 (1998), 338–341.

[Ki99] S. A. Kirillov, *Norm estimates of functions in Lorentz spaces*, Acta Sci. Math. (Szeged) 65 (1999), 189–201.

[Kol92] V. I. Kolyada, *Some generalizations of the Hardy–Littlewood–Paley theorem*, Mat. Zametki 51 (1992), 24–34; English transl.: Math. Notes 51 (1992), 235–244.

[Pa31] R. E. A. C. Paley, *Some theorems on orthogonal functions* I, Studia Math. 3 (1931), 226–238.

[Ri23] F. Riesz, *Über eine Verallgemeinerung der Parsevalschen Formel*, Math. Z. 18 (1923), 117–124.

[Ro01] T. V. Rodionov, *Analogues of the Hausdorff–Young and Hardy–Littlewood theorems*, Izv. Ross. Akad. Nauk Ser. Mat. 65 (2001), no. 3, 175–192; English transl.: Izv. Math. 65 (2001), 589–606.

[SW58] E. M. Stein, G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), 159–172.

One of the classical orthonormal systems of functions is the *Haar system* $\{h_n\}_{n=1}^{\infty}$ in which the functions are defined on the unit interval $[0,1]$ in the following way: $h_1(x) = 1$ for all $x \in [0,1]$ and if $n = 2^m + k$, $k = 1, 2, \dots, 2^m$, $m = 0, 1, \dots$, then

$$h_n(x) = \begin{cases} 2^{m/2}, & \text{if } x \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -2^{m/2}, & \text{if } x \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & \text{if } x \notin (\frac{k-1}{2^m}, \frac{k}{2^m}). \end{cases}$$

At interior points of discontinuity a Haar function is put equal to half the sum of its limiting values from the right and from the left, and at the end points of $[0,1]$ to its limiting values from within the interval.

Schauder [Sc28] proved that Haar system is a basis (Schauder basis) in $L^p[0,1]$ for $1 \le p < \infty$. Marcinkiewicz continued studies of the Haar system by showing in [M37a] that it is an unconditional basis for $1 < p < \infty$.

THEOREM 27 (Marcinkiewicz 1937). *For $1 < p < \infty$ the Haar system is an unconditional basis in $L^p[0,1]$, that is, it remains a basis under any permutation of its elements.*

This theorem is sometimes called the Paley–Marcinkiewicz theorem since Marcinkiewicz's proof is a consequence of Paley's results on the Walsh system. Gundy [Gu67], Burkholder [Bu73] and Gapoškin [Ga74] gave simple proofs of this theorem. A generalization of Marcinkiewicz's Theorem 27 on reflexive Orlicz spaces was presented by Gapoškin [Ga67], [Ga68], and on separable symmetric spaces X on $[0,1]$ with Boyd indices $0 < \alpha_X \le \beta_X < 1$ by Semenov [Se69] (this theorem with the proof can be found in the books [KPS82], pp. 181–182 and [LT79], pp. 156–158).

More information and the proof of Theorem 27 can be found in the books and papers cited below:

[KS58] S. Kaczmarz, H. Steinhaus, *Theory of Orthogonal Series*, Fizmatgiz, Moscow, 1958 (Russian) [Marcinkiewicz theorems, pp. 449–450].

[Si70] I. Singer, *Bases in Banach Spaces I*, Grundlehren Math. Wiss. 154, Springer, Berlin, 1970 [Theorem 14.1. Marcinkiewicz theorem, pp. 407–409 and 633].

[Ol75] A. M. Olevskiĭ, *Fourier Series with Respect to General Orthogonal Systems*, Ergeb. Math. Grenzgeb. 86, Springer, New York, 1975 [Marcinkiewicz, p. 71].

[LT79] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*, Springer, Berlin–New York, 1979 [Theorem 2.c.5, pp. 155–156].

[KPS82] S. G. Krein, Yu. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian); English transl.: Transl. Math. Monogr. 54, Amer. Math. Soc., Providence, 1982 [Theorem 9.6. for the L_p spaces was obtained by Marcinkiewicz, pp. 181–182 and 351].

[KS84] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (Russian) [Theorem 3.8. Marcinkiewicz theorem, pp. 92].

[NS97] I. Novikov, E. Semenov, *Haar Series and Linear Operators*, Math. Appl. 367, Kluwer Acad. Publ., Dordrecht, 1997 [Marcinkiewicz theorem, pp. 36–39].

[Bu73] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), 19–42.

[Ga67] V. F. Gaposhkin, *The existence of unconditional bases in Orlicz spaces*, Funkcional. Anal. i Prilozhen. 1 (1967), no. 4, 26–32; English transl.: Functional Anal. Appl. 1 (1967), 278–284.

[Ga68] V. F. Gaposhkin, *Unconditional bases in Orlicz spaces*, Sibirsk. Mat. Zh. 9 (1968), 280–287; English transl.: Siberian Math. J. 9 (1968), 211–217.

[Ga74] V. F. Gaposhkin, *The Haar system as an unconditional basis in $L_p[0,1]$*, Mat. Zametki 15 (1974), 191–196; English transl.: Math. Notes 15 (1974), 108–111.

[Go70] B. I. Golubov, *Series in the Haar system*, in: Mathematical Analysis 1970, Itogi Nauki, Akad. Nauk SSSR VINITI, Moscow, 1971, 109–146; English transl.: J. Soviet Math. (New York) 1 (1973), 704–726.

[Gu67] R. F. Gundy, *The martingale version of a theorem of Marcinkiewicz and Zygmund*, Ann. Math. Statist. 38 (1967), 725–734.

[GU58] R. S. Guter, P. L. Ul'yanov, *On some results in the theory of orthogonal series*, supplement to the translation from German into Russian of the book: S. Kaczmarz, H. Steinhaus, *Theorie der Orthogonalreihen*, Warszawa–Lwów 1935, GIFML, Moscow, 1958 (Russian) [Marcinkiewicz theorem on Haar functions, pp. 449–450]; English transl.: *Supplement to theory of orthogonal series*, Amer. Math. Soc. Transl. II. Ser. 17 (1961), 219–250.

[Ha10] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. 69 (1910), 331–371.

[Kro78] V. G. Krotov, *Unconditional convergence of the Fourier series in the Haar system in the spaces Λ^p_ω*, Mat. Zametki 23 (1978), 685–695; English transl.: Math. Notes 23 (1978), 376–382.

[Pa32] R. E. A. C. Paley, *A remarkable series of orthogonal functions* I, II, Proc. London Math. Soc. (2) 34 (1932), 241–264 34 (1932), 265–279.

[Pe85] A. Pełczyński, *Norms of classical operators in function spaces*, Astérisque 131 (1985), 137–162 [2. The Marcinkiewicz–Paley inequality for the Haar system, pp. 144–146].

[Sc28] J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. 28 (1928), 317–320.

[Se69] E. M. Semenov, *A certain method for obtaining interpolation theorems in symmetric spaces*, Dokl. Akad. Nauk SSSR 185 (1969), 1243–1246; English transl.: Soviet Math. Dokl. 10 (1969), 507–511.

[Ul61] P. L. Ul'yanov, *Divergent Fourier series*, Uspekhi Mat. Nauk 16 (1961), no. 3, 61–142 [Theorem D. Marcinkiewicz result, pp. 68–69]; English transl.: Russian Math. Surveys 16 (1963), no. 3, 1–74.

[Wa91] G. Wang, *Sharp square-function inequalities for conditionally symmetric martingales*, Trans. Amer. Math. Soc. 328 (1991), 393–419.

4.5. Approximation theory. We discussed earlier the approximation of functions by Fourier series. Now, we instead concentrate on Lagrange type interpolation and trigonometric interpolation.

4.5.1. *Marcinkiewicz's theorem on Lagrange interpolation.* The Lagrange interpolation problem is to construct, for a given continuous function $f : [a,b] \to \mathbb{R}$ and different n points (nodes) $x_1, x_2, \ldots, x_n$ from the interval $[a,b]$, a polynomial $L_{n-1}(f,x)$ of degree at most $n-1$ such that $L_{n-1}(f,x_k) = f(x_k)$ for $k = 1,2,\ldots,n$. A polynomial of this type can be found by using Lagrange's interpolation formula. Then also an estimate on the error $f(x) - L_{n-1}(f,x)$ is of interest.

An interpolation process is given on $[a,b]$ if there is given an infinite triangular matrix of nodes (table of nodes) $(x_{n,k})_{1\le k\le n, 1\le n}$ such that in every row we have different points $x_{n,k}\neq x_{n,j}$ for $k\neq j$ from interval $[a,b]$.

We do not necessarily have $L_{n-1}(f,x)\to f(x)$, but S. Bernstein constructed (for certain interpolation points) other interpolation polynomials $A_n(f,x)$ of degree $m_n>n$ such that $A_n(f,x)\to f(x)$. G. Faber (1914) has shown that there is no table of nodes that the corresponding interpolation process is uniformly convergent for any continuous function, since for any table of nodes on $[-1,1]$ there is an $f\in C[-1,1]$ for which a sequence of Lagrange polynomials $L_{n-1}(f,x)$ satisfies $\limsup_{n\to\infty}|L_{n-1}(f,x)|=\infty$. On the other hand, we have Marcinkiewicz's theorem concerning the possibility to construct a table of nodes for every function separately (cf. [M36e], Thm 3).

THEOREM 28 (Marcinkiewicz 1936). *For any continuous function f on $[a,b]$ there exists an infinite triangular matrix of nodes such that the corresponding interpolation process for f is uniformly convergent.*

Theorem 28 can be found, for example, in the book by Daugavet [Da77], pp. 153–154.

4.5.2. *Grünwald–Marcinkiewicz interpolation theorem.* Given $2n+1$ distinct points $x_0,x_1,\dots,x_{2n}$ on the x-axis, mod 2π, and any $2n+1$ real numbers $y_0,y_1,\dots,y_{2n}$, there exists a trigonometric polynomial

$$T_n(x)=\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

of degree n (it has $2n+1$ coefficients) such that $T_n(x_k)=y_k$ $(k=0,1,\dots,2n)$ and such a polynomial is unique. The most interesting case is when the points $e^{ix_0},e^{ix_1},\dots,e^{ix_{2n}}$ are equally distributed on the unit circle, that is, when $x_k=x_0+\frac{2\pi k}{2n+1}$, and the numbers $y_0,y_1,\dots,y_{2n}$ are the values of a continuous function $f(x)$ at the points $x_0,x_1,\dots,x_{2n}$. The polynomial $T_n(x)=T_n(f,x)$ is then called the *Lagrange interpolating polynomial* for the function f; it depends also on the point x_0, but for the sake of simplicity we may fix the point x_0 once for all and take $x_0=0$. Polynomials T_n are given by the formulas

$$T_nf(x)=\frac{1}{\pi}\int_0^{2\pi}f(t)\,\frac{\sin(n+\frac{1}{2})(x-t)}{2\sin\frac{1}{2}(x-t)}\,d\omega_{2n+1}(t),\tag{59}$$

where $\omega_{2n+1}(t)$ is a step function having jumps $2\pi/(2n+1)$ at the points x_k and is continuous elsewhere. If we replace here $d\omega_{2n+1}(t)$ by dt we obtain the classical formula for the nth partial sums $S_nf(x)$ of the Fourier series of the function f. Since when $n\to\infty$ the graph of $\omega_{2n+1}(t)$ approaches, after the subtraction of a suitable constant, the limit $g(t)=t$, it is natural to conjecture that the behaviour of the sequence (T_n) is for $n\to\infty$ similar to the behaviour of the sequence (S_n).

It is actually so within certain limits. For example, already Faber showed that there exists a continuous function f such that the sequence $(S_nf(x))$ is uniformly convergent while the sequence $(T_n(f,x))$ diverges at certain individual points. In 1933, Marcinkiewicz in his PhD [M35b] constructed *a continuous function f such that the sequence $(S_nf(x))$ converges uniformly while the sequence $(T_n(f,x))$ diverges almost everywhere.*

At the time when the dissertation was published (as a paper [M35c]), the paper [Gr35] of the Hungarian mathematician Géza Grünwald[13] appeared. It contains a similar result for the so-called Tchebyshev interpolation, which differs only formally from the Lagrange interpolation. The matrix of nodes in Tchebyshev interpolation is given by points $\{x_{n,k} = \cos \frac{2k-1}{2n}\pi\}_{1\le k\le n, 1\le n}$ on the interval $[-1,1]$, and an interpolation polynomial by $L_{n-1}(f,x)$.

It is curious that a year later both authors could, independently of each other, strengthen their examples by constructing continuous functions whose Tchebyshev interpolating polynomials diverge everywhere.

THEOREM 29 (Grünwald 1936, Marcinkiewicz 1936/37). *There exists a function* $f \in C[-1,1]$ *such that the sequence* $(L_{n-1}(f,x))_{n=1}^{\infty}$ *is divergent at every point* $x \in [-1,1]$, *i.e.,* $\limsup_{n\to\infty} |L_{n-1}(f,x)| = \infty$ *for any* $x \in [-1,1]$.

The proof of Theorem 29 is given, for example, in the books of Zygmund ([Zy59], Vol. II, pp. 44–46) and Natanson [Na55]. The topic was developed later by G. Grünwald, A. A. Privalov, P. Turán, P. Erdős and P. Vértesi. In particular, the last two authors have shown in [EV80] the Grünwald–Marcinkiewicz theorem for any arrangement of nodes.

Theorem 29 is important for two reasons. On the one hand it shows that the Lagrange approximation method is sometimes not a good approximation, even at the nodes. On the other hand, we can see many similarities between approximation by the Fourier sums $S_n f$ and the Lagrange polynomial interpolation $L_{n-1}(f,\cdot)$. These similarities could be used, although to a limited extent. For example, Marcinkiewicz in his master thesis noticed that a *continuous function* f *for which the sequence* $(T_n f(x))$ *diverges almost everywhere* (*or even everywhere*) *can satisfy the condition* $f(x+h) - f(x) = O(1/\log(1/h))$, *i.e., the sequence* $(S_n f(x))$ *is convergent almost everywhere.*

If we replace O by o the sequence $(T_n f(x))$ converges uniformly. Another example, is the Carleson theorem (1964), which shows that if $f \in C[-1,1]$, then the partial Fourier sums $S_n f(x)$ converge to $f(x)$ almost everywhere on $[-1,1]$, but from Theorem 28 this is not the case for the sequence $(T_n f(x))$. Marcinkiewicz proved in [M38i] that if $f \in L^1$ is periodic and F is the indefinite integral of the function f, then the derivatives of $T_n'(F,x)$ converge to $f(x)$ and better imitate the behaviour of the partial sums $S_n f(x)$ than the polynomials $T_n(f,x)$.

It is also worth to mention about the following result of Marcinkiewicz from 1936 ([M36b], Thm 1): *there is a continuous* 2π*-periodic function* f *such that the arithmetic means*

$$\frac{T_0 f(x) + T_1 f(x) + \ldots + T_n f(x)}{n+1}$$

diverge at some points. This means that we do not have an obvious analogue of the classical theorem of Fejér about the arithmetic means of the partial sums of Fourier series.

[13] Géza Grünwald (born 18 October 1910 in Budapest – killed 7 September 1942, as holocaust victim).

This theorem, proofs and generalizations can be found in the following books and papers:

[Na55] I. P. Natanson, *Konstruktive Funktionentheorie*, Akademie, Berlin, 1955 [II.3. Ein Beispiel von Marcinkiewicz, pp. 379–388].

[Ch66] E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966 [information on Marcinkiewicz (1937) and Grünwald (1936) result, p. 233].

[Da75] P. J. Davis, *Interpolation and Approximation*, Dover, New York, 1975 [Marcinkiewicz, p. 79].

[Da77] I. K. Daugavet, *Introduction to the Theory of Approximation of Functions*, Izdat. Leningrad. Univ., Leningrad, 1977 (Russian) [Chapter 5, Theorem 1 (Marcinkiewicz), pp. 153–154].

[SV90] J. Szabados, P. Vértesi, *Interpolation of Functions*, World Scientific, Singapore, 1990 [Grünwald–Marcinkiewicz result, p. 126].

[Ti94] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Dover, New York, 1994 [information on Marcinkiewicz and Grünwald result, p. 579].

[EV80] P. Erdős, P. Vértesi, *On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes*, Acta Math. Acad. Sci. Hungar. 36 (1980), 71–89.

[Gr35] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome*, Acta Sci. Math. (Szeged) 7 (1935), 207–221.

[Gr36] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen*, Ann. of Math. (2) 37 (1936), 908–918.

[Gr43] G. Grünwald, *On the theory of interpolation*, Acta Math. 75 (1943), 219–245.

[MV01] T. M. Mills, P. Vértesi, *An extension of the Grünwald–Marcinkiewicz interpolation theorem*, Bull. Austral. Math. Soc. 63 (2001), 299–320.

[Of40] A. C. Offord, *Approximation to functions by trigonometric polynomials*, Duke Math. J. 6 (1940), 505–510.

[Re03] M. Revers, *A survey on Lagrange interpolation based on equally spaced nodes*, in: Advanced Problems in Constructive Approximation (Dortmund, 2001), Birkhäuser, Basel, 2003, 153–163.

4.5.3. *Marcinkiewicz–Zygmund inequalities.* In the proofs on the convergence of Lagrange interpolation, there are some inequalities used in the estimation error of Lagrange interpolation – they compare a continuous norm and its discretization. We are talking about the results originating in papers of Marcinkiewicz [M36b] and Marcinkiewicz–Zygmund [MZ37a].

THEOREM 30 (Marcinkiewicz inequalities 1936). *If $1 < p < \infty$, then there exist constants $A_p, B_p > 0$ such that for arbitrary trigonometric polynomial T of degree $\leq n$ the following inequalities hold:*

$$\frac{A_p}{2n+1}\sum_{k=0}^{2n}\Big|T\Big(\frac{2k\pi}{2n+1}\Big)\Big|^p \leq \int_0^{2\pi} |T(x)|^p\,dx \leq \frac{B_p}{2n+1}\sum_{k=0}^{2n}\Big|T\Big(\frac{2k\pi}{2n+1}\Big)\Big|^p.$$

THEOREM 31 (Marcinkiewicz–Zygmund inequalities 1937). *If $1 < p < \infty$, then there exist constants $C_p, D_p > 0$ such that for any complex polynomial P of degree $\leq n$ we have* *the following inequalities*

$$\frac{C_p}{n+1}\sum_{k=0}^{n}\left|P(e^{2\pi ki/(n+1)})\right|^p \le \int_0^{2\pi} |P(e^{i\theta})|^p\,\frac{d\theta}{2\pi} \le \frac{D_p}{n+1}\sum_{k=0}^{n}\left|P(e^{2\pi ki/(n+1)})\right|^p.$$

Note that for $p = 1$ and $p = \infty$ the first inequalities in Theorems 30 and 31 are still true, but the second ones are not true. Proofs of the inequalities in Theorem 30 can be found in Marcinkiewicz's paper ([M36b], Thms 9 and 10) and Marcinkiewicz–Zygmund's paper ([MZ37a], Thms 1 and 2), and also in the book by Zygmund ([Zy59], Theorem 7.5). Proofs of Theorem 31 are in the paper by Marcinkiewicz and Zygmund ([MZ37a], Thm 10) and the Zygmund's book ([Zy59], Theorem 7.10).

The above inequalities have been generalized in different ways, see the papers cited below.

[Mi70] D. S. Mitrinović, *Analytic Inequalities*, Grundlehren Math. Wiss. 165, Springer, Berlin, 1970 [3.5.35. Marcinkiewicz and Zygmund inequalities, p. 261].

[Ku04] J. C. Kuang, *Applied Inequalities*, 3rd ed., Shangdong 2004 (Chinese) [42. Marcinkiewicz–Zygmund inequalities, p. 432].

[BKP09] A. Böttcher, S. Kunis, D. Potts, *Probabilistic spherical Marcinkiewicz–Zygmund inequalities*, J. Approx. Theory 157 (2009), 113–126.

[CZ99] C. K. Chui, L. Zhong, *Polynomial interpolation and Marcinkiewicz–Zygmund inequalities on the unit circle*, J. Math. Anal. Appl. 233 (1999), 387–405.

[Da03] S. B. Damelin, *Marcinkiewicz–Zygmund inequalities and the numerical approximation of singular integrals for exponential weights: methods, results and open problems, some new, some old*, J. Complexity 19 (2003), 406–415.

[DJK02] S. B. Damelin, H. S. Jung, K. H. Kwon, *Converse Marcinkiewicz–Zygmund inequalities on the real line with application to mean convergence of Lagrange interpolation*, Analysis (Munich) 22 (2002), 33–55.

[Lu97] D. S. Lubinsky, *Marcinkiewicz–Zygmund inequalities: methods and results*, in: Recent Progress in Inequalities (Niš, 1996), Math. Appl. 430, Kluwer, Dordrecht, 1998, 213–240.

[Lu99] D. S. Lubinsky, *On converse Marcinkiewicz–Zygmund inequalities in L_p, $p > 1$*, Constr. Approx. 15 (1999), 577–610.

[Ma07] J. Marzo, *Marcinkiewicz–Zygmund inequalities and interpolation by spherical harmonics*, J. Funct. Anal. 250 (2007), 559–587.

[MR00] G. Mastroianni, M. G. Russo, *Weighted Marcinkiewicz inequalities and boundedness of the Lagrange operator*, in: Mathematical Analysis and Applications, Hadronic Press, Palm Harbor, 2000, 149–182.

[MN01] H. N. Mhaskar, F. J. Narcowich, J. D. Ward, *Spherical Marcinkiewicz–Zygmund inequalities and positive quadrature*, Math. Comp. 70 (2001), 1113–1130.

[MP98] H. N. Mhaskar, J. Prestin, *On Marcinkiewicz–Zygmund-type inequalities*, in: Approximation Theory (in memory of A. K. Varma), Monogr. Textbooks Pure Appl. Math. 212, Marcel Dekker, New York, 1998, 389–404.

[OS07] J. Ortega-Cerdà, J. Saludes, *Marcinkiewicz–Zygmund inequalities*, J. Approx. Theory 145 (2007), 237–252.

[RS98] K. V. Runovskii, H. J. Schmeisser, *On Marcinkiewicz–Zygmund type inequalities for irregular knots in L_p-spaces,* $0 < p \le +\infty$, Math. Nachr. 189 (1998), 209–220.

[Sc08] D. Schmid, *Marcinkiewicz–Zygmund inequalities and polynomial approximation from scattered data on* SO(3), Numer. Funct. Anal. Optim. 29 (2008), 855–882.

[Xu91] Y. Xu, *The generalized Marcinkiewicz–Zygmund inequality for trigonometric polynomials*, J. Math. Anal. Appl. 161 (1991), 447–456.

[Xu91c] Y. Xu, *On the Marcinkiewicz–Zygmund inequality*, in: Progress in Approximation Theory, Academic Press, Boston, 1991, 879–891.

[ZS94] L. Zhong, X. Shen, *Weighted Marcinkiewicz–Zygmund inequalities*, Adv. in Math. (China) 23 (1994), 66–75.

4.6. Some other Marcinkiewicz's results. Here we present six other themes in which there are results of Marcinkiewicz, and which were cited in some books and papers.

4.6.1. *Strong summability of Fourier series.* From the Fejér theorem, for every $f \in L^1$ we have convergence almost everywhere of $\sigma_n f(x) \to f(x)$, when $n \to \infty$, i.e.

$$\frac{1}{n+1} \sum_{m=0}^{n} [S_m f(x) - f(x)] = o(1) \quad \text{a.e.}$$

Hardy and Littlewood (1927) asked for the truth of the following stronger property

$$\frac{1}{n+1} \sum_{m=0}^{n} |S_m f(x) - f(x)| = o(1) \quad \text{a.e.},$$

and then for strong summability H_r in the sense of Hardy of order $r > 0$, i.e., the property

$$\frac{1}{n+1} \sum_{m=0}^{n} |S_m f(x) - f(x)|^r = o(1) \quad \text{a.e.} \tag{60}$$

By the Hölder–Rogers inequality we can easily see that for larger r the result is stronger.

In 1939 Marcinkiewicz [M39d] gave a positive answer to the problem of Hardy–Littlewood in the case $r = 2$, proving that: *if $f \in L^1$, then*

$$\frac{1}{n+1} \sum_{m=0}^{n} [S_m f(x) - f(x)]^2 = o(1) \quad \text{a.e.}$$

This theorem with a proof is e.g. in the book of Bary ([Ba64], II, pp. 24–31). Zygmund [Zy42] generalized this theorem to any $r > 0$ in the property (60) and his proof is completely different (cf. also Zygmund book [Zy59], II, pp. 185–186).

[Zy42] A. Zygmund, *On the convergence and summability of power series on the circle of convergence.* II, Proc. London Math. Soc. (2) 47 (1942), 326–350.

[Ta55] K. Tandori, *On strong summability of Fourier series*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 457–465 (Hungarian).

4.6.2. *Absolutely convergent Fourier series.* A theorem of P. Lévy, generalizing a theorem of N. Wiener (see Zygmund [Zy59], I, p. 245 and Bary [Ba64], II, pp. 190–194), states that if $g(x)$ has an absolutely convergent Fourier series, and if $f(x)$ is analytic in the closed interval $(\min g(x), \max g(x))$, then $f[g(x)]$ also has an absolutely convergent Fourier series. Marcinkiewicz showed in [M40] that this result can be extended by requiring less on $f(x)$ and more on $g(x)$.

For $0 < s \leq 1$ and an open interval $I \subset \mathbb{R}$ denote by $G_s(I)$ the class of infinitely differentiable functions F on I satisfying inequalities of the type $|F^{(n)}(x)| \leq B^n n^{n/s}$ on every closed subinterval of I. For $0 < p < \infty$ denote by A_p the class of functions $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$ such that $\sum_{n\in\mathbb{Z}} |a_n|^p < \infty$. Marcinkiewicz proved that: *if* $f(t) \in A_s$ $(0 < s \leq 1)$, *I contains the range of f and* $F \in G_s(I)$, *then* $F(f) \in A_1$. Zygmund has pointed out that the proof of Marcinkiewicz can be extended to show that in fact $F(f) \in A_s$. This result is called the *Wiener–Lévy–Marcinkiewicz theorem* on absolutely convergent series. Rivière and Sagher [RS66] proved the converse of Marcinkiewicz's theorem, in stronger form. Marcinkiewicz's method was used by Kahane, Katznelson, Mallivan and others in the so-called "symbolic calculus" in or between the algebras A_p. More information can be found in [Ka68] and [Ka70].

The interest in the space A_p for $0 < p < 1$ follows from the fact that this is a nontrivial example of locally bounded algebra which is not a Banach algebra.

Marcinkiewicz also showed that ([M40]; see also Bary [Ba64], II, pp. 194–196): *there exist functions $f(x)$ and $g(x)$, both having absolutely convergent Fourier series, but such that the Fourier series of $f[g(x)]$ does not converge absolutely. The function $f(x)$ is equal to zero in $(-\pi, 0)$ and at π, equal to $(\log x)^{-2}$ in $(0, \frac{1}{2})$, and linear in $(\frac{1}{2}, \pi)$; and $g(x) = f(x)$.*

[Ka70] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergeb. Math. Grenzgeb. 50, Springer, Berlin–New York, 1970 [VI.4. Method of Marcinkiewicz, pp. 77–80]; Russian transl.: Mir, Moscow, 1976 [VI. 4. Method of Marcinkiewicz, pp. 96–99].

[Dy71] E. M. Dyn'kin, *Individual theorems of Wiener–Levy type for Fourier series and integrals*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 22 (1971), 181–182 (Russian).

[Ka68] J.-P. Kahane, *Sur les séries de Fourier à coefficients dans l^p*, in: Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press, Carbondale, Ill., 1968, 257–272.

[RS66] N. M. Rivière, Y. Sagher, *The converse of Wiener–Levy–Marcinkiewicz theorem*, Studia Math. 28 (1966), 133–138.

[Ul02] P. L. Ul'yanov, *On Lévy and Marcinkiewicz theorems for Fourier–Haar series*, Izv. Nats. Akad. Nauk Armenii Mat. 36 (2001), no. 4, 73–81. English transl.: J. Contemp. Math. Anal. 36 (2001), no. 4, 77–85.

4.6.3. *Thin sets related to trigonometric series.* A set $E \subset [0, 2\pi]$ is called a *set of uniqueness*, or a *U-set*, if any trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ which converges to zero for $x \notin E$ is identically zero; that is, such that $a_n = b_n = 0$ for all n. Otherwise E is a *set of multiplicity* (sometimes called an *M-set* or a *Menshov set*). If $E \subset [0, 2\pi]$ is an M-set, then there is a trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ that converges to zero on $[0, 2\pi] \setminus E$ and that has nonzero coefficients. Analogous definitions apply on the real line, and in higher dimensions. Every countable set is a U-set. Every set E of positive measure is an M-set.

Marcinkiewicz and Zygmund proved that ([MZ37d]; see also Bary [Ba64], II, p. 364): *If E is a U-set and $\theta > 0$ is such that $E(\theta) = \{\theta x : x \in E\} \subset [0, 2\pi]$, then $E(\theta)$ is a U-set.* For example, for $0 < \theta < 1$ the set $E(\theta)$ is again a U-set.

Marcinkiewicz [M38f], in honour of V. V. Nemytzkii, introduced the notion of an

N-set (investigated earlier by P. Fatou (1906) and A. Rajchman (1922)): a set $A \subset [0,1]$ is an *N-set* if there is a trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$ which converges absolutely in A but not everywhere, that is, $\sum_{n=1}^{\infty}(|a_n| + |b_n|) = \infty$. Equivalently (see [BKR95], pp. 467–468), a set $A \subseteq [0,1]$ is an N-set if and only if there are non-negative reals $(\rho_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \rho_n = \infty$ such that the series $\sum_{n=1}^{\infty} \rho_n \sin \pi nx$ absolutely converges for $x \in A$; if and only if there are reals $\rho_n \geq 0$, $k_n \geq 1$, $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and the series $\sum_{n=1}^{\infty} \rho_n \sin k_n x$ converges for $x \in E$. Every countable set is of type N. If E is of type N and D countable, then $E + D$ is of type N. Marcinkiewicz [M38d] proved that we cannot replace D by an arbitrary set of type N, that is, the sum of two N-sets is not an N-set in general. The sum of two sets of type N may be the whole interval $(0, 2\pi)$, and so not of type N.

THEOREM 32 (Marcinkiewicz 1938). *There exist two sets A and B of type N such that $A + B$ is not of type N.*

A proof of this theorem can be found, e.g., in the Zygmund book ([Zy59], I, pp. 238–239) and Bary book ([Ba64], II, pp. 305–306). Arbault gave in [Ar52] a simple proof of this theorem.

[KS63] J.-P. Kahane, R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1963; 2nd ed., Paris, 1994.

[Ar52] J. Arbault, *Sur l'ensemble de convergence absolue d'une série trigonométrique*, Bull. Soc. Math. France 80 (1952), 253–317.

[BKR95] L. Bukovský, N. N. Kholshchevnikova, M. Repický, *Thin sets of harmonic analysis and infinite combinatorics*, Real Anal. Exchange 20 (1994/95), 454–509.

[Ka02] J.-P. Kahane, *Sets of uniqueness and sets of multiplicity*, in: Fourier Analysis and Related Topics, Banach Center Publ. 56, Warsaw, 2002, 55–68 [4. Zygmund and Marcinkiewicz, p. 58].

[Sa41] R. Salem, *On some properties of symmetrical perfect sets*, Bull. Amer. Math. Soc. 47 (1941), 820–828.

[Ul02a] P. L. Ul'yanov, *On interconnections between the research of Russian and Polish mathematicians in the theory of functions*, in: Fourier Analysis and Related Topics, Banach Center Publl. 56, Warsaw, 2002, 119–130.

4.6.4. *The Lévy–Raikov–Marcinkiewicz theorem on analytic properties of characteristic functions.* The following principle is well known in harmonic analysis: if μ is a positive finite measure and its Fourier transform $\hat{\mu}(t) = \int_{\mathbb{R}} e^{-itx}\, d\mu(x)$ is "smooth" at the origin, then it is "smooth" on the whole real line.

P. Lévy (1937) and D. Raikov (1938) (see, e.g., [LO77], Theorem 2.2.1, p. 24) showed that if the Fourier transform $\hat{\mu}$ coincides in the real neighbourhood $(-a, a)$ of the origin with a function which is analytic in a rectangle $\{z : |\operatorname{Re} z| < a,\ -R < \operatorname{Im} z < R\}$, then $\hat{\mu}$ admits analytic continuation to the strip $\{z : |\operatorname{Im} z| < R\}$. As a generalization of the real analyticity in $(-a, a) \subset \mathbb{R}$, one can consider weaker property of a function g to be the boundary value of a function which is analytic in a complex upper half-neighbourhood of $(-a, 0)$:

(A) g coincides in a real neighbourhood $(-a, a)$ of the origin with a function which is analytic in a rectangle $\{z : |\operatorname{Re} z| < a,\ 0 < \operatorname{Im} z < R\}$ and continuous in its closure.

Marcinkiewicz [M38e] (see also [LO77], Theorem 2.2.3, p. 25) showed that the principle also works with this generalized real analyticity. The result can be stated in the following form:

The Lévy–Raikov–Marcinkiewicz theorem (1937–1938): *if μ is a finite nonnegative Borel measure whose Fourier transform $\hat{\mu}$ satisfies assumption (A), $\hat{\mu}$ admits analytic continuation into the strip $\{z : 0 < \operatorname{Im} z < R\}$ and is representable there by the absolutely convergent integral.*

The extension of this result to general classes of measures and distributions, assuming non-negativity only on some half-line $(b, +\infty)$ were given by Ostrovskii and Ulanovskii in [OU03] and [OU04].

[Ka72] T. Kawata, *Fourier Analysis in Probability Theory*, Probability and Math. Statist. 15, Academic Press, New York–London, 1972 [theorem given by Marcinkiewicz, pp. 456–457].

[LO77] Ju. V. Linnik, I. V. Ostrovskiĭ, *Decomposition of Random Variables and Vectors*, Transl. Math. Monogr. 48, Amer. Math. Soc., Providence, 1977 [Theorem 2.2.3. Marcinkiewicz theorem, pp. 25–27 and 360].

[OU03] I. Ostrovskii, A. Ulanovskii, *On the Lévy–Raikov–Marcinkiewicz theorem*, C. R. Math. Acad. Sci. Paris 336 (2003), 237–240.

[OU04] I. Ostrovskii, A. Ulanovskii, *On the Lévy–Raikov–Marcinkiewicz theorem*, J. Math. Anal. Appl. 296 (2004), 314–325.

4.6.5. *The circular structure of the set of limit points of partial sums of Taylor series.* For a complex power series $\sum a_n z^n$, let $S_n(z) = \sum_{k=1}^{n} a_k z^k$, $s_n(x) := S_n(e^{ix})$ for $x \in \mathbb{R}$, and $\sigma_n(x) = \frac{s_0(x)+\ldots+s_n(x)}{n+1}$ (the $(C,1)$-mean). For each x, let $L(x)$ denote the set of limit points of $\{\sigma_n(x)\}$. Marcinkiewicz and Zygmund [MZ41] proved the following theorem (see also Zygmund [Zy59], II, pp. 178–179): *If $E := \{x : \lim_{n\to\infty} \sigma_n(x) = \sigma(x)$ exists$\}$ then, for almost all $x \in E$ and for every $\alpha \in L(x)$, the whole circumference $\{z : |z - \sigma(x)| = |\alpha - \sigma(x)|\}$ is included in $L(x)$.*

The question of the angular equidistribution of $\{s_n(x)\}$ almost everywhere in E was considered by Kahane [Ka83]. Examples suggest that not only the limit points of $\{\sigma_n(x)\}$ but the partial sums $s_n(x)$ themselves lie on $L(x)$, and Katsoprinakis [Kat89] proved a general result in this direction when E is uncountable. A related result by Katsoprinakis and Nestoridis [KN89] also answers a question posed by Kahane. A final result is due to Nestoridis [Ne92] and concerns power series $\sum c_k z^k$ with $\liminf |c_k| > 0$ and E an infinite subset of the unit circle (see also [NP90]). In the paper [KP99] there is an extension of Marcinkiewicz–Zygmund's $(C,1)$ result into (C,k) summability with $k \geq 1$.

[Ka83] J.-P. Kahane, *Sur la structure circulaire des ensembles de points limites des sommes partielles d'une série de Taylor*, Acta Sci. Math. (Szeged) 45 (1983), 247–251.

[Kat89] E. S. Katsoprinakis, *On a theorem of Marcinkiewicz and Zygmund for Taylor series*, Ark. Mat. 27 (1989), 105–126.

[KN89] E. S. Katsoprinakis, V. N. Nestoridis, *Partial sums of Taylor series on a circle*, Ann. Inst. Fourier (Grenoble) 39 (1989), 715–736.

[KP99] E. S. Katsoprinakis, M. Papadimitrakis, *Extensions of a theorem of Marcinkiewicz–Zygmund and of Rogosinski's formula and an application to universal Taylor series*, Proc. Amer. Math. Soc. 127 (1999), 2083–2090.

[Ne92] V. Nestoridis, *Limit points of partial sums of Taylor series*, Mathematika 38 (1991), 239–249.

[NP90] V. Nestoridis, S. K. Pichorides, *The circular structure of the set of limit points of partial sums of Taylor series*, in: Séminaire d'Analyse Harmonique, Année 1989/90, Univ. Paris XI, Orsay, 1990, 71–77.

4.6.6. *Correction theorem.* Let $X_0 \subset X$ be two function spaces on $[0,1]$ with the Lebesgue measure m. Recall that X_0 is said to *correct* X if for every $f \in X$ and any $\varepsilon > 0$ there exists a function $g = g_\varepsilon \in X_0$ such that $m(\{t \in [0,1] : f(t) \neq g(t)\}) < \varepsilon$. The classical N. N. Luzin theorem (1912) on C-property of a measurable function $f : [a,b] \to \mathbb{R}$ states that for any $\varepsilon > 0$ there exists a closed set $F \subset [0,1]$ such that $m([0,1] \setminus F) < \varepsilon$ and $f_{|F}$ is continuous. This means that $C[0,1]$ corrects $L^0[0,1]$-measurable functions on $[0,1]$. In other words, we can correct a measurable function on arbitrary small set in such a way that the function is continuous. Marcinkiewicz showed that ([M36a], Thm 3): $C^1[0,1]$ *corrects* $\mathrm{Lip}^1[0,1]$. Marcinkiewicz, in fact, assumed "pointwise" Lip^1, that is, $f(x+t) = f(x) + O(t)$ for any x and showed similar results for higher order smoothness. Federer [Fe44] obtained the analogous Lip^1-C^1 result in higher dimension, and Whitney [Wh51] extended Federer's result to higher order of smoothness. He also gave an example of one variable function $\phi \in \mathrm{Lip}^\alpha[0,1]$ for any $0 < \alpha < 1$ for which the conclusion in Marcinkiewicz's theorem is not true. Since we have $C^1 \subset \mathrm{Lip}^1 \subset \bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha$, then Whitney example means that one could not weaken in the Marcinkiewicz theorem the requirement of f being in Lip^1 by $f \in \bigcap_{0<\alpha<1} \mathrm{Lip}^\alpha$. In [BK03] the authors showed that the Takagi–van der Waerden function is also such an example.

[BK03] J. B. Brown, G. Kozlowski, *Smooth interpolation, Hölder continuity, and the Takagi–van der Waerden function*, Amer. Math. Monthly 110 (2003), no. 2, 142–147.

[Fe44] H. Federer, *Surface area II*, Trans. Amer. Math. Soc. 55 (1944), 438–456.

[Ki84] S. V. Kislyakov, *A new correction theorem*, Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), 305–330. English transl.: Math. USSR.-Izv. 24 (1985), 283–306.

[Ki95] S. V. Kislyakov, *A sharp correction theorem*, Studia Math. 113 (1995), 177–196.

[Wh51] H. Whitney, *On totally differentiable and smooth functions*, Pacific J. Math. 1 (1951), 143–159.

Marcinkiewicz proved also many other theorems, but they did not have that much resonance as those presented previously.

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The photos enclosed here are coming from the Marcinkiewicz's family collections (photos 2, 8, 10, 11), Archive of the Institute of Mathematics Polish Academy of Sciences in Sopot (photos 1, 6, 15), Central Lithuanian Archive in Vilnius (photos 3, 4, 5, 7), Library of the Nicolaus Copernicus University in Toruń (photos 9, 14). I want to thank all of them for permission to put these photos here. Photos 12 and 13 I did myself in 2010.

List of publications of Józef Marcinkiewicz[14]

[M33] *A new proof of a theorem on Fourier series*, J. London Math. Soc. 8 (1933), 179 [JMCP, 35].

[M34] *On a class of functions and their Fourier series*, C. R. Soc. Sci. Varsovie 26 (1934), 71–77 [JMCP, 36–41].

[M35a] *Sur les nombres dérivés*, Fund. Math. 24 (1935), 305–308 [JMCP, 42–44].

[M35b] *Wielomiany interpolacyjne funkcji bezwzględnie ciągłych* [*Interpolating polynomials for absolutely continuous functions*], PhD Thesis, Universitas Vilnensis Batorena, Facultas Scientiarum–Dissertationes Inaugurales, No. 10, Warszawa, 1935, 1–35 (Polish) (a copy of this dissertation is, for example, in Archive [MJ1]).

[M35c] *Wielomiany interpolacyjne funkcji bezwzględnie ciągłych* [*Interpolating polynomials for absolutely continuous functions*], Wiadom. Mat. 39 (1935), 85–125 (Polish); English transl.: [JMCP, 45–70].

[M35d] *On the convergence of Fourier series*, J. London Math. Soc. 10 (1935), 264–268 [JMCP, 71–75].

[M35e] *On Riemann's two methods of summation*, J. London Math. Soc. 10 (1935), 268–272 [JMCP, 76–80].

[MJZ35] (and B. Jessen, A. Zygmund) *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), 217–234 [JMCP, 81–95].

[M36a] *Sur les séries de Fourier*, Fund. Math. 27 (1936), 38–69 [JMCP, 96–124].

[MZ36] (and A. Zygmund) *On the differentiability of functions and summability of trigonometrical series*, Fund. Math. 26 (1936), 1–43 [JMCP, 125–163].

[M36b] *Sur l'interpolation I*, Studia Math. 6 (1936), 1–17 [JMCP, 171–185].

[M36c] *Sur la convergence des séries orthogonales*, Studia Math. 6 (1936), 39–45 [JMCP, 164–170].

[M36d] *Sur l'interpolation II*, Studia Math. 6 (1936), 67–81 [JMCP, 186–199].

[14]In square brackets the pages are given on which the paper has been reprinted or translated in [JMCP] = *Józef Marcinkiewicz, Collected Papers*, Edited by Antoni Zygmund, PWN, Warsaw, 1964, viii+673 pp.

[M36e] *Quelques remarques sur l'interpolation*, Acta Litt. Sci. Szeged, Sect. Sci. Math. 8 (1936/37), 127–130 [JMCP, 200–203].

[M36f] *Sur la divergence des polynomes d'interpolation*, Acta Litt. Sci. Szeged, Sect. Sci. Math. 8 (1936/37), 131–135 [JMCP, 204–208].

[MZ37a] (and A. Zygmund) *Mean values of trigonometrical polynomials*, Fund. Math. 28 (1937), 131–166 [JMCP, 233–259].

[MZ37b] (and A. Zygmund) *Some theorems on orthogonal systems*, Fund. Math. 28 (1937), 309–335 [JMCP, 209–232].

[MZ37c] (and A. Zygmund) *Sur les fonctions indépendantes*, Fund. Math. 29 (1937), 60–90 [JMCP, 233–259].

[MZ37d] (and A. Zygmund) *Two theorems on trigonometric series*, Rec. Math. Moscou (Mat. Sbornik) 2(44) (1937), 733–737 [JMCP, 293–298].

[MZ37e] (and A. Zygmund) *Remarque sur la loi logarithme itéré*, Fund. Math. 29 (1937), 215–222 [JMCP, 299–306].

[M37a] *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. 16 (1937), 84–96 [JMCP, 307–318].

[M37b][15] *Quelques théorèmes sur les séries et les fonctions*, Trav. Soc. Sci. Lettres Wilno Cl. Sci. Math. Nat. 12 (1937), 1–6.

[M38a] *O sumowalności szeregów ortogonalnych* [*On the summability of orthogonal series*], Wiadom. Mat. 44 (1938), 5–16 (Polish); English transl.: [JMCP, 319–327].

[M38b] *Sur les fonctions indépendantes I*, Fund. Math. 30 (1938), 202–214 [JMCP, 328–340].

[M38c] *Sur les fonctions indépendantes II*, Fund. Math. 30 (1938), 349–364 [JMCP, 341–354].

[M38d] *Sur les suites d'opérations linéaires*, Studia Math. 7 (1938), 52–72 [JMCP, 355–373].

[MK38] (and S. Kaczmarz) *Sur les multiplicateurs des séries orthogonales*, Studia Math. 7 (1938), 73–81 [JMCP, 389–396].

[MZ38a] (and A. Zygmund) *Quelques théorèmes sur les fonctions indépendantes*, Studia Math. 7 (1938), 104–120 [JMCP, 374–388].

[M38e] *Sur les fonctions indépendantes III*, Fund. Math. 31 (1938), 86–102 [JMCP, 397–412].

[M38f] *Quelques théorèmes sur les séries et les fonctions*, Bull. Sémin. Math. Univ. Wilno 1 (1938), 19–24 [JMCP, 413–417].

[M38g] *Quelques théorèmes sur les séries orthogonales lacuinaires*, Ann. Soc. Polon. Math. 17 (1938), 51–58 [JMCP, 418–423].

[MZ38b] (and A. Zygmund) *Proof of a gap theorem*, Duke Math. J. 4 (1938), 469–472 [JMCP, 424–427].

[MZ38c] (and A. Zygmund) *A theorem of Lusin*, Duke Math. J. 4 (1938), 473–485 [JMCP, 428–443].

[M38h] *Sur quelques intégrales du type de Dini*, Ann. Soc. Polon. Math. 17 (1938), 42–50 [JMCP, 444–451].

[MZ38d] (and A. Zygmund) *Sur les séries de puissances*, Mathematica (Cluj) 14 (1938), 21–30 [JMCP, 454–462].

[15] R. Salem [Sa41] (see references in Part 4.6.3) referred to this Marcinkiewicz paper [M37b] which is not on the list of publications published in [JMCP]. Moreover, A. Zygmund in his review of Salem's paper in Math. Reviews MR0005132 (3,105f) added even pages on which Marcinkiewicz's article was published and this information was not in the Salem's article [Sa41]. Marcinkiewicz's paper [M37b] is also cited in the articles [Ar52] and [BKR95]. The paper has identical title as Marcinkiewicz's paper [M38f], but it is not clear if they are the same or different papers.

[M38i] *Un théorème sur l'interpolation*, Mathematica (Cluj) 14 (1938), 36–38 [JMCP, 452–453].

[M39a] *Sur une propriété de la loi de Gauss*, Math. Z. 44 (1939), 612–618 [JMCP, 463–469].

[M39b] *Une remarque sur les espaces de M. Besicovitch*, C. R. Acad. Sci. Paris 208 (1939), 157–159 [JMCP, 470–472].

[M39c] *Sur le problème des moments*, C. R. Acad. Sci. Paris 208 (1939), 405–407 [JMCP, 473–474].

[M39d] *Sur la sommabilité H_k des séries de Fourier*, C. R. Acad. Sci. Paris 208 (1939), 782–784 [JMCP, 475–477].

[M39e] *Sur les séries orthogonales*, Studia Math. 8 (1939), 1–27 [JMCP, 478–500].

[M39f] *Sur les multiplicateurs des séries de Fourier*, Studia Math. 8 (1939), 78–91 [JMCP, 501–512].

[M38–40] *Sur une propriété du mouvement brownien*, Acta Litt. Sci. Szeged 9 (1938–40), 77–87 [JMCP, 513–523].

[MB39] (and S. Bergmann) *Sur les valeurs limites des fonctions de deux variables complexes*, C. R. Acad. Sci. Paris 208 (1939), 877–879 [JMCP, 524–526].

[M39g] *Sur une méthode remarquable de sommation des séries doubles de Fourier*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) 8 (1939), 149–160 [JMCP, 527–538].

[M39h] *Sur l'interpolation d'opérations*, C. R. Acad. Sci. Paris 208 (1939), 1272–1273 [JMCP, 539–540].

[MZ39a] (and A. Zygmund) *Quelques inéqualitiés pour les opérations linéaires*, Fund. Math. 32 (1939), 115–121 [JMCP, 541–546].

[MZ39b] (and A. Zygmund) *On the summability of double Fourier series*, Fund. Math. 32 (1939), 122–132 [JMCP, 547–556].

[M39i] *Sur les variables aléatoires enroulées*, C. R. Soc. Math. France, année 1938(1939), 34–36 [JMCP, 557–558].

[M39j] *Sur la sommabilité forte de séries de Fourier*, J. London Math. Soc. 14 (1939), 162–168 [JMCP, 559–565].

[M39k][16] *Quelques théorèmes de la théorie des probabilités*, Bull. Sém. Math. Univ. Wilno 2 (1939), 22–34 [JMCP, 566–579].

[M39l] *Sur une nouvelle condition pour la convergence presque partout des séries de Fourier*, Ann. Scuola Norm. Super. Pisa (2) 8 (1939), 239–240 [JMCP, 580–581].

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