ON AN EXTREMAL PROPERTY OF CHEBYSHEV POLYNOMIALS

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Given a closed interval $S = [a, b]$ of length $\ell = b - a$, and two positive numbers $\lambda = \theta\ell$, $0 < \theta < 1$, and $0 < \kappa$, we consider the following problem[1]:

Problem. *Find an exact upper bound on the quantity*

$$\max_{a \le x \le b} |P_n(x)| \tag{1}$$

where P_n is a polynomial of degree at most n satisfying the inequality

$$|P_n(x)| \le \kappa \tag{2}$$

on a set of points (otherwise undetermined) $E \subset S$ of measure $\ge \lambda$.

We will show that the upper bound in question has the exact value

$$M = \kappa\, T_n\left(\frac{2\ell}{\lambda} - 1\right) = \kappa\, T_n\left(\frac{2}{\theta} - 1\right) \tag{3}$$

where T_n is the trigonometric polynomial of degree n

$$T_n(z) = \frac{1}{2}\left\{(z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n\right\}. \tag{4}$$

Solution. We will first verify that (1) attains the value (3) for the two Chebyshev polynomials

$$P_{n,1}(x) = \kappa\, T_n\left(\frac{2x - a - (a + \lambda)}{\lambda}\right) \tag{5}$$

and

$$P_{n,2}(x) = \kappa\, T_n\left(\frac{2x - (b - \lambda) - b}{\lambda}\right), \tag{6}$$

[1] The author encountered this problem in the course of his work on the convergence of a certain process of successive approximations that he had proposed for the effective calculation of the polynomial of best approximation to a bounded function $f(x)$ on a uniformly bounded set of points. (Cf. my note, Comptes Rendus, Paris, 30, VII, 1934 and also my article in Ukrainian: “On the methods for realizing the best approximation of functions in the sense of Chebyshev”, Acad. des Sc. d’Ukraine, 1935, pp. 99–100).

which satisfy the condition (2), one on the interval $[a, a+\lambda]$, the other on the interval $[b-\lambda, b]$. It remains to prove that among all admissible polynomials $P_n(x)$, the two polynomials (5) and (6) are the only (up to multiplication by ± 1), for which the quantity (1) attains the value (3).

Let $P_n(x)$ be an *admissible* polynomial different from (5) and (6); let $E \subset S$ be a set of points on which (2) holds. This set of points is evidently composed of a certain number $\nu \leq n$ of closed intervals some which can be one point. Let

$$\sigma_1 = [\alpha_1, \beta_1], \quad \sigma_2 = [\alpha_2, \beta_2], \dots, \sigma_m = [\alpha_m, \beta_m] \tag{7}$$

be those of them ($m \leq \nu$) of positive length arranged in increasing order. Let $\xi \in S$ be a point such that $|P_n(x)|$ attains its maximum value on the interval $[a, b]$:

$$|P_n(\xi)| = \max_{a \leq x \leq b} |P_n(x)|. \tag{8}$$

We will show that $|P_n(\xi)| \leq M$, where M designates the value (3).

We distinguish between three cases depending on:

$$\xi > \beta_m, \quad \xi < \alpha_1 \text{ or finally } \beta_i < \xi < \alpha_{i+1} \tag{9}$$

where in the last case $i \in \{1, 2, \dots, m-1\}$.

We start by considering *the first case*. Let $x_1 = a, x_2, x_3, \dots, x_{n+1} = a+\lambda$ be the points on the interval $[a, a+\lambda]$, where the Chebyshev polynomial (5) attains, with alternating sign, the values $\pm\kappa$. Let, in addition, $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{n+1}$ be the $n+1$ points that we take in the set E satisfying the following conditions: firstly, $\overline{x}_1 = \alpha_1$; then for $\ell = 2, 3, \dots, n+1$ let $\overline{x}_\ell$ be the first of the points in E (traversing this set of points from left to right) for which

$$\text{mes}([\overline{x}_1, \overline{x}_\ell] \cdot E) = x_\ell - x_1, \tag{10}$$

the product in the parenthesis meaning the set of points appearing both in the interval $[\overline{x}_1, \overline{x}_\ell]$ and in the set E. {Transl: The intersection of the two sets.}

Applying the Lagrange interpolation formula, one time with the polynomial (5) and another time with the polynomial $P_n(x)$, we can write the following two equalities:

$$M = P_{n,1}(b) = \sum_{i=1}^{n+1} \frac{(b-x_1)\cdots(b-x_{i-1})(b-x_{i+1})\cdots(b-x_{n+1})}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{n+1})} P_{n,1}(x_i) \tag{11}$$

$$P_n(\xi) = \sum_{i=1}^{n+1} \frac{(\xi-\overline{x}_1)\cdots(\xi-\overline{x}_{i-1})(\xi-\overline{x}_{i+1})\cdots(\xi-\overline{x}_{n+1})}{(\overline{x}_i-\overline{x}_1)\cdots(\overline{x}_i-\overline{x}_{i-1})(\overline{x}_i-\overline{x}_{i+1})\cdots(\overline{x}_i-\overline{x}_{n+1})} P_n(\overline{x}_i). \tag{12}$$

On comparing their right-hand parts term by term, one notes the following relations:

α) $\quad |P_{n,1}(x_i)| = \kappa; \ |P_{n,1}(\overline{x}_i)| \leq \kappa$

β) $\quad b - x_j \geq \xi - \overline{x}_j \geq 0$

γ) $\quad |x_i - x_j| \leq |\overline{x}_i - \overline{x}_j|, \qquad i, j = 1, 2, \dots, n+1; \ j \neq i.$

Moreover, one also sees that the $n+1$ terms on the last part of (11) *are all the same sign* (being +), which need not hold in (12). Thus one also has

$$|P_n(\xi)| < M,$$

at least that $P_n(x)$ is not identical to $\pm P_{n,1}(x)$.

In the *second case* (9), that is to say when $\xi < \alpha_1$, the reasoning is totally analogous, on replacing the polynomial (5) by (6).

Finally in the case in (9)

$$\beta_i < \xi < \alpha_{i+1} \tag{14}$$

set

$$\frac{\text{mes}([a,\xi]\cdot E)}{\xi - a} = \theta_1, \tag{15}$$

$$\frac{\text{mes}([\xi,b]\cdot E)}{b - \xi} = \theta_2\,. \tag{16}$$

It is clear that the two numbers θ_1 and θ_2 can not be *at the same time* less than $\theta = \frac{\lambda}{\ell}$. Replacing, in the previous reasoning, the interval $[a,b]$ once by $[a,\xi]$ and another time by $[\xi,b]$, one has *simultaneously*

$$\begin{aligned} |P_n(\xi)| &< \kappa T_n\left(\frac{2}{\theta_1} - 1\right) \\ |P_n(\xi)| &< \kappa T_n\left(\frac{2}{\theta_2} - 1\right)\,, \end{aligned} \tag{17}$$

and one of the right hand sides above is certainly $\leq M$ and the proof is complete.

We have simultaneously obtained a simple proof of a known theorem of Chebyshev[2] which derives from our reasoning when *a priori* restricting the field of admissible polynomials[3].

[2] P. L. Tchebyshef, "Sur les fonctions qui s'écartent peu de zéro pour certaines valeurs de la variable", Œuvres tome 2, pp. 335–355.

[3] As understood, one sets *a priori* $E = [a, a+\lambda]$ or $E = [b-\lambda, b]$.