FUNCTIONS OF REAL VARIABLES.

Leçons sur les Fonctions de Variables réelles et les Développements en Séries de Polynomes. Professées a l'École normale supérieure par ÉMILE BOREL, et rédigées par MAURICE FRÉCHET. Avec des notes par PAUL PAINLEVÉ et HENRI LEBESGUE. Paris, Gauthier-Villars, 1905. viii + 160 pp.

THE theory of functions has for several years been distinctly the fashionable thing in mathematical circles in Paris, and the publication of Borel's collection of monographs on the subject, of which the volume now under review is the eighth, aims to gain devotees elsewhere. For we take the chief purpose of this series of publications to be the encouragement of research by the presentation in connected form of the more recent results in the theory of functions, each volume confining itself to some limited domain of investigation. So the present volume confines itself in the main to a connected exposition of the present state of the theory of series of polynomials of a real variable, connected not merely in itself, but also with the older theory.

This volume, like its predecessors, is characterized by great precision of statement, rigor of proof, and elegance of treatment throughout. If a single word were desired to describe the whole, we could think of no better than "modern." The book is fairly alive with the modern spirit. As an instance we may note the fact that the so-called Heine-Borel theorem, which the reviewer believes should play a fundamental rôle in every modern treatment of the theory of functions of real variables, is introduced at the outset (page 9) and is used extensively throughout the volume, greatly simplifying many proofs and adding largely to the general elegance of the exposition.

As has already been said, one of the characteristics of the volume is precision of statement. It is strange, therefore, that one of the few criticisms that we have to make should be for a lack of precision. On page 6 the author speaks of "the segment (0, 1), which we will call the *fundamental interval.*" We are not told however that the end points are to be included. On the following page it becomes evident that 0 and 1 are points of the fundamental interval; and in fact throughout the volume the symbol (a, b) seems to be used for the set of points x satisfying the condition $a \leqq x \leqq b$ (in this review we will so understand the symbol), but this is nowhere stated explicitly, and the occurrence of the phrase "the interval (a, b), extremities included" (page 27, *e. g.*) seems to imply that the last two qualifying words are necessary. This we believe is a lack of precision in notation. It would be better surely to introduce at the outset unique symbols to denote intervals with end points included, and with one or both excluded.*

As may be inferred from the above, the first of the five chapters that make up the body of the volume concerns itself with the theory of point sets. Many of the definitions and theorems here given have of course been known for some time. And yet ideas of recent date are introduced almost at once and with telling effect in the simplification of older proofs. Thus the theorem of Cantor-Bendixson that any closed set can be resolved into a perfect set and an enumerable set, which was originally proved by the aid of transfinite numbers, receives a simple elementary proof by the use of Lindelöf's notion of a point of condensation.

The notion of the measure of a point set (in the Borel sense) receives extended treatment. It is fundamental in much of what follows. What the author calls the fundamental theorem of measure, viz., that the points of an enumerable set of intervals whose total length is less than unity cannot exhaust all the points of an interval of length one, follows almost immediately from the Heine-Borel theorem, already referred to. The general notion of measure is introduced in practically the same way as in the author's earlier volume, Leçons sur la théorie des fonctions (1898). This method of treatment is in so far unsatisfactory that it does not furnish a general means of determining whether or not a given point set is measurable. However, it is possible to show that every point set which is explicitly defined by certain very general operations is measurable. The author, in fact, goes further. He claims that any set which can be explicitly constructed (que nous pourrons *effectivement* former) must be definable in terms of the operations referred

* Peano has used the symbols $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, to denote respectively all points x satisfying the conditions $a \leqq x \leqq b$, $a \leqq x < b$, $a < x \leqq b$, $a < x < b$. (Cf. Lezioni di analisi infinitesimale, vol. 1 (1893), p. 9.) More recently Pierpont, Theory of functions of real variables, vol. 1, p. 119, has suggested the symbols (a, b), (a, b^*), (a^*, b), (a^*, b^*) to denote the same sets of points. Moreover, the two words "interval" and "segment" may well be used to distinguish between inclusion and exclusion of end points, the former, say, being used when the end points are included, the latter if one or both of them are excluded. Cf. Veblen, "On the Heine-Borel theorem," BULLETIN, vol. 9 (1904), p. 487.

to, and hence that every set that we can explicitly define is measurable.

Continuity is the subject of the second chapter. The definitions are the usual ones. Some of the fundamental theorems receive more or less obvious generalizations. The theorem, for example, that a function which is continuous in an interval (a, b) is uniformly continuous in the interval, is stated more generally as follows, $\omega(x)$ denoting the oscillation at the point x:

If in the interval (a, b) *we have* $\omega(x) \leqq c$, *we can associate with every positive number* ϵ *a positive number* η, *such that the inequality* $|\xi_1 - \xi_2| < \eta$ *implies, for any pair of points* ξ_1, ξ_2 *of* (a, b), *the inequality* $|f(\xi_1) - f(\xi_2)| < c + \epsilon$.*

The proof by means of the Heine-Borel theorem is very simple. The chapter closes with an elegant discussion of Lebesgue's generalization of the notion of integral.

Chapter III. deals with series of real functions, the bulk of the chapter being devoted to a very clear discussion of the conditions under which a convergent series of continuous functions defines a continuous function. The solution of the problem of deriving the necessary and sufficient condition herefor is due to Arzelà. The condition in question, for which Arzelà introduced the term "uniform convergence by segments," is called "quasi-uniform convergence" by M. Borel. Townsend † has used Moore's term "subuniform convergence" as an equivalent, in his recent article on Arzelà's condition; it would seem to be the most convenient. The definition is stated as follows, $r_n(x)$ denoting the remainder after n terms:

A series is said to converge subuniformly in (a, b), *if* 1) *the series converges in* (a, b); 2) *it is possible to associate with every* ϵ, *and every* N, *a finite* $N' > N$, *such that for every* x *in* (a, b) *there exists an integer* n_x *lying between* N *and* N' *such that* $|r_{n_x}(x)| < \epsilon$.

Arzelà's theorem states *that the necessary and sufficient condition that a series whose terms are continuous in* (a, b) *shall define a continuous function in that interval, is that the series shall converge subuniformly in the interval.* This theorem was first given by Arzelà in 1884, but the proof was attacked on the score of rigor. His later proofs (1899 and 1902) which he claims are

*In the original b is used where we have used c, thus introducing ambiguity in the use of b.

† BULLETIN, vol. 12 (1905), p. 17. Cf. also Moore, BULLETIN, vol. 7 (1901), p. 257.

sound are complicated. Under these conditions one must needs admire the extreme simplicity of the proof given by Borel in this volume. It occupies barely two pages.*

The point of departure in the next (fourth) chapter, which deals with the representation of continuous functions by series of polynomials, is naturally the theorem of Weierstrass to the effect that every continuous function can be so represented. Several proofs of this theorem are given, including the original of Weierstrass, that of Volterra, and the purely elementary ones of Runge, Mittag-Leffler and Lebesgue. We find also the extension to functions of several variables, and Borel's theorem on the representation of a continuous function, which has continuous derivatives of every order in the interval $(-1, 1)$ as the sum of a Fourier series and a power series in x. The last eighteen pages of the chapter are devoted to the application of the foregoing theory to the theory of interpolation. A critical examination of Lagrange's formula shows that it can not be relied upon to give an approximation to the given function which increases as the number of known values increases. It is possible, however, *to define once for all polynomials* $P_{p,q}(x)$ *such that any function* $f(x)$, *continuous in* $(0, 1)$, *can be represented by the series, uniformly convergent in* $(0, 1)$,

$$f(x) = \Pi_1 + (\Pi_2 - \Pi_1) + \cdots + (\Pi_n - \Pi_{n-1}) + \cdots,$$

where

$$\Pi_q = \sum_{p=0}^{p=q} f\left(\frac{p}{q}\right) P_{p,q}(x).$$

This is the general formula of interpolation which Borel presented in the summer of 1904 to the Third International Congress of Mathematicians. His procedure is beautifully simple and is briefly as follows: †

Let $\phi_{p,q}(x)$ be a continuous function which in the interval $(0, 1)$ coincides with the function represented by the broken line $OMPNA$ in the figure where P is the point $(p/q, 1)$. It is then clear that if x_1 is any point of the interval $((p-1)/q, p/q)$ and if $\phi_{p,q}(x_1) = \theta$, then $\phi_{p-1,q}(x_1) = 1 - \theta$, and all the other

* We may refer the reader who is interested in the comparison to the account of Arzela's work in the paper by Townsend already mentioned, which also contains several interesting examples illustrative of the theory.

† Borel's treatment is throughout purely analytic. We have given it this geometric phrasing merely for the sake of greater brevity of statement.

$\phi_{i,q}(x_1)$ vanish. Now, if q be taken so large that the inequality $|x_1 - x_2| < 1/q$ insures the inequality $|f(x_1) - f(x_2)| < \epsilon$ throughout the interval (0, 1), then it follows immediately that

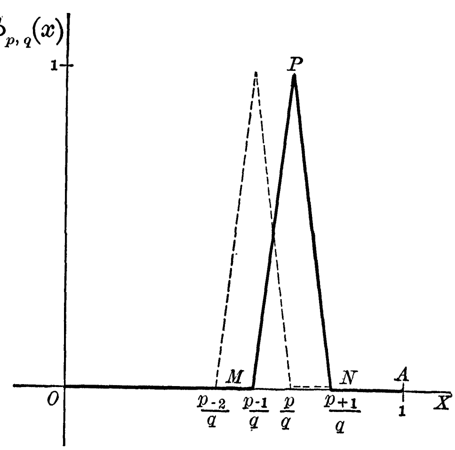
$$\left|\sum_{p=0}^{p=q} f\left(\frac{p}{q}\right)\phi_{p,q}(x) - f(x)\right| < \epsilon$$

throughout (0, 1). It is then only necessary to take for $P_{p,q}(x)$ a polynomial such that the inequality

$$|\phi_{p,q}(x) - P_{p,q}(x)| < 1/q^2$$

holds in (0, 1), to obtain Borel's theorem.

This result is theoretically of much interest. But the reviewer can not agree with M. Borel, when he says in the preface that



this formula is "susceptible of almost immediate application to practical problems of great variety." Any practical application must depend on the explicit calculation of the polynomials $P_{p,q}(x)$, and the nature of the function $\phi_{p,q}(x)$ would seem to require the degree of these polynomials to be very high,* too

* A method for calculating these polynomials has recently been published (M. Potron, "Sur une formule générale d'interpolation," *Bulletin de la Société mathématique de France*, vol. 34 (1906), p. 52) which makes the degree of $P_{p,q}(x)$ equal to $4q^6$. This number may well be capable of reduction by the use of another method for the calculation; but it is hardly open to question that the polynomial of *lowest* degree satisfying the conditions will still be of little value for practical purposes.

high in any case to be of practical value. In this connection the following modification of Borel's procedure may be of interest. His argument will hold word for word if his function $\phi_{p,q}(x)$ be replaced by any continuous function $\psi_{p,q}(x)$, which satisfies the following conditions: (1) it shall vanish for all points of (0, 1), except for points of the segment $[(p-1)/q, (p+1)/q]$, and (2) for all values of x in the interval $[(p-1)/q, p/q]$ it shall satisfy the relation

$$\psi_{p,q}(x) + \psi_{p,q}(x + 1/q) = 1.$$

The function $\phi_{p,q}(x)$ is theoretically the simplest satisfying these conditions, but it is quite possible that it is not the most convenient for the purpose of representation by means of a polynomial. It would be interesting to determine, if possible, the polynomial $Q_{p,q}(x)$ of lowest degree satisfying the following conditions:

$$1)\ |Q_{p,q}(x)| < \frac{1}{q^2} \text{ if } 0 \leqq x \leqq \frac{p-1}{q} \text{ or } \frac{p+1}{q} \leqq x \leqq 1\ (p<q),$$

$$2)\ \left|Q_{p,q}(x) + Q_{p,q}\left(x + \frac{1}{q}\right) - 1\right| < \frac{1}{q^2}, \text{ if } \frac{p-1}{q} \leqq x \leqq \frac{p}{q}.$$

This suggests at once a possible generalization of the problem of approximation usually associated with the name of Tchebicheff, where the given function is replaced by an arbitrary function which is required merely to satisfy certain functional relations. The ordinary Tchebicheff problem, *i. e.*, to find that polynomial of given degree which approximates most closely to a given continuous function in a given interval, is given a masterly exposition * in the concluding pages of the chapter.

The eight pages of the fifth and last chapter give a brief introduction to the methods and results recently developed by Baire concerning the representation of discontinuous functions by series of polynomials.

In addition to the five chapters of M. Borel's monograph the volume contains three notes; two short ones by Lebesgue and Borel on "A proof of a theorem by M. Baire" and "On the existence of functions of any class whatever" respectively, and

* On page 87, there is an error of detail; η must be chosen so that $|Q(x)|$ is not only less than $\mu - \varepsilon$, but also less than $\mu - \mu'$.

one of 48 pages by Painlevé on "The development of analytic functions." The latter will perhaps be to some the most interesting part of the book, as in it M. Painlevé gives for the first time a connected development of the results which he has recently published in the *Comptes Rendus* concerning the expansion of analytic functions in series that are valid in the whole complex plane with the exception only of certain half-lines, *i. e.*, in the whole "star of holomorphism." But this review is already too long; we can not go into details.

The following are the only typographical errors we have noticed; page 7, line 2, for F read P, and line 8, for M read N; page 16, line 5 from bottom, for E_1 read E_2, and line 4 from bottom, for E_2 read E_1; page 33, line 9, after "limites" insert "inférieures et"; page 58, line 8 in the formula replace y_i by $(y_i - y_{i-1})$; page 60 in the expansion for $\sqrt{x^2}$ the last factors in the numerators of the coefficients of the second and third terms should be removed; page 80, the equations defining $\phi_{p,q}(x)$ are not correct when $p = 0$; page 82, in the last sentence of the footnote for Kircherberger read Kirchberger; page 85, line 15 from bottom, for A read A'.

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PRINCETON UNIVERSITY,
March 23, 1906.

SHORTER NOTICES.

Advanced Algebra. By HERBERT E. HAWKES. Ginn and Company, 1905. 285 pp.

IN writing an Advanced Algebra, in the current acceptation of the phrase, two courses are fairly open to an author. He may lay his foundations deep and build on them with unflinching rigor, bringing teacher and pupil into intimate touch with some of the epoch-making researches of the past fifty years. From the standpoint of pure science, this is, of course, admirable; but the meat which it supplies is only for strong men. Foundation-laying is for the hardiest of frontiersmen; and the methodical account of its severely logical steps makes demands upon the average freshman which are far beyond his power to meet.

The second course consists in a frank and explicit assumption of the postulates, as they become necessary in the order of